On kinematic waves

I. Flood movement in long rivers

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In this paper and in part II, we give the theory of a distinctive type of wave motion, which arises in any one-dimensional flow problem when there is an approximate functional relation at each point between the flow q (quantity passing a given point in unit time) and concentration k (quantity per unit distance). The wave property then follows directly from the equation of continuity satisfied by q and k. In view of this, these waves are described as 'kinematic', as distinct from the classical wave motions, which depend also on Newton's second law of motion and are therefore called 'dynamic'. Kinematic waves travel with the velocity $\partial q/\partial k$, and the flow q remains constant on each kinematic wave. Since the velocity of propagation of each wave depends upon the value of q carried by it, successive waves may coalesce to form 'kinematic shock waves'. From the point of view of kinematic wave theory, there is a discontinuous increase in q at a shock, but in reality a shock wave is a relatively narrow region in which (owing to the rapid increase of q) terms neglected by the flow-concentration relation become important. The general properties of kinematic waves and shock waves are discussed in detail in §1. One example included in §1 is the interpretation of the group-velocity phenomenon in a dispersive medium as a particular case of the kinematic wave phenomenon.

The remainder of part I is devoted to a detailed treatment of flood movement in long rivers, a problem in which kinematic waves play the leading role although dynamic waves (in this case, the long gravity waves) also appear. First (§2), we consider the variety of factors which can influence the approximate flow-concentration relation, and survey the various formulae which have been used in attempts to describe it. Then follows a more mathematical section (§3) in which the role of the dynamic waves is clarified. From the full equations of motion for an idealized problem it is shown that at the 'Froude numbers' appropriate to flood waves, the dynamic waves are rapidly attenuated and the main disturbance is carried downstream by the kinematic waves; some account is then given of the behaviour of the flow at higher Froude numbers. Also in §3, the full equations of motion are used to investigate the structure of the kinematic shock; for this problem, the shock is the 'monoclinal flood wave' which is well known in the literature of this subject. The final sections (§§4 and 5) contain the application of the theory of kinematic waves to the determination of flood movement. In §4 it is shown how the waves (including shock waves) travelling downstream from an observation point may be deduced from a knowledge of the variation with time of the flow at the observation point; this section then concludes with a brief account of the effect on the waves of tributaries and run-off. In §5, the modifications (similar to diffusion effects) which arise due to the slight dependence of the flow-concentration curve on the rate of change of flow or concentration, are described and methods for their inclusion in the theory are given.

1. Introduction

In this paper and in part II (Lighthill & Whitham 1955), we wish to draw attention to a class of wave motions physically quite distinct from the classical wave motions encountered in dynamical systems. They have received some attention already in connexion with flood movement in long rivers, but no general treatment seems to have been given.

The waves will be considered only for one-dimensional flow systems. Then they exist if, to sufficient approximation, there is a functional relationship between

(i) the flow q (quantity† passing a given point in unit time),
(ii) the concentration k (quantity per unit distance), and
(iii) the position x.

On this assumption the wave property follows from the equation of continuity alone. Accordingly, we suggest that the waves be described as 'kinematic'. The classical wave motions would in contrast be described as 'dynamic' waves, depending as they do on Newton's second law of motion, together with some assumption relating a stress to a displacement (as in gravity waves), to a strain (as in the non-dispersive longitudinal and transverse waves), or to a curvature (as in capillary waves and flexural waves).

One important difference is that kinematic waves possess only one wave velocity at each point, while dynamic waves possess at least two (forwards and backwards relative to the medium). This is because the equation of continuity, that is, the conservation law

$$\frac{\partial k}{\partial t}+\frac{\partial q}{\partial x}=0, \tag{1}$$

which states that the quantity in a small element of length changes at a rate equal to the difference between the inflow and outflow, is of the first order only. If we assume that

$$q=q(k,x), \tag{2}$$

then, on multiplying (1) by

$$c=\left(\frac{\partial q}{\partial k}\right)_{x\ \text{constant}}=c(k,x), \tag{3}$$

we obtain

$$\frac{\partial q}{\partial t}+c\frac{\partial q}{\partial x}=0. \tag{4}$$

This means that q is constant on waves travelling past the point with velocity c given by (3). Mathematically, the equation has one system of 'characteristics' (given by $\mathrm{d}x=c\,\mathrm{d}t$), and along each of these the flow q is constant.

The wave velocity c, by (3), is the slope of the flow-concentration curve for fixed x. This fact has been referred to in the literature on flood movements (see § 2 below) as the Kleitz–Seddon law.

In terms of the mean velocity at a point, which is

$$v=q/k, \tag{5}$$

the wave velocity is

$$c=\frac{\mathrm{d}}{\mathrm{d}k}(vk)=v+k\frac{\mathrm{d}v}{\mathrm{d}k}. \tag{6}$$

Thus $c>v$ when the mean velocity increases with concentration (as in rivers), while $c<v$ when it decreases with concentration (as in traffic flow).

Kinematic waves are not dispersive, but they suffer change of form due to non-linearity‡ (dependence of the wave velocity c on the flow q carried by the wave) exactly as do travelling sound waves of finite amplitude. Accordingly, continuous

† For example, volume of water (in a river), number of vehicles (on a road).
‡ This might be called 'amplitude dispersion', in contrast to 'frequency dispersion'.

wave forms may develop discontinuities, due to the overtaking of slower waves by faster ones. We propose to describe these as shock waves, since their process of formation is exactly that of shock waves in a gas.

The law of motion of kinematic shock waves is derived from conservation considerations, as was the law governing continuous kinematic waves. If the flow and concentration take the values q_1, k_1 on one side, and q_2, k_2 on the other side, of the shock wave, which moves with speed U, then the quantity crossing it per unit time may be written either as $q_1 - Uk_1$ or as $q_2 - Uk_2$. This gives the velocity of the shock wave as

$$U = \frac{q_2 - q_1}{k_2 - k_1}. \tag{7}$$

This is the slope of the chord joining the two points on the flow-concentration curve (for given x) which correspond to the states ahead of and behind the shock wave when it reaches x. In the limit when the shock wave becomes a continuous wave, the slope of the chord becomes the slope of the tangent and the velocity given by (7) coincides with that given by (3).

It will appear that kinematic shock waves can change strength by absorbing continuous waves, and can unite with other shock waves to form single shock waves, exactly like dynamic shock waves in gases.

Now there is probably no system in which the flow, as has been assumed, is accurately a function of concentration and position. Normally some small time lag may intervene between adjustments of flow and concentration at a given point; or, again, the relationship between them may have only statistical validity. Treatment as a 'kinematic wave' will describe the development of the flow with reasonable accuracy over times large compared with such a time lag, provided that the diffusive effects due to it, and to statistical deviations from the mean flow-concentration relation, are small by comparison with the wave effects. Estimates of accuracy from such considerations are obtained below and in part II.

In particular, the shock waves will not be perfect discontinuities. They will have a definite thickness, produced (as with dynamic shock waves in gases) by a balance between the diffusive effects mentioned above† and the tendency to thinning due to the excess wave velocity behind the shock wave over that in front. However, it may still be convenient to calculate the motion of these shock waves as if they were discontinuous, bearing in mind their real thickness when the theory is finally interpreted.

The formation of a kinematic shock wave is illustrated in figures 1, 2 and 3, in the specially simple and important case when the flow-concentration curve (figure 1) is independent of the position x, that is, when q is a function of k alone. In this case, since q is constant along any wave, k and hence c must also be constant along it, so the wave moves with constant velocity. Thus, in a space-time diagram (figure 2), the waves are straight lines, parallel to the tangent to the flow-concentration curve

† The diffusive effects may be represented (§ 5) by a second derivative of q inserted in the equation of motion (4). The thickness of a shock wave is governed by this higher-order term in the equation, which outside the shock wave is negligible—exactly as in gas dynamics, where the analogous higher-order term represents the effects of viscosity and heat conduction.

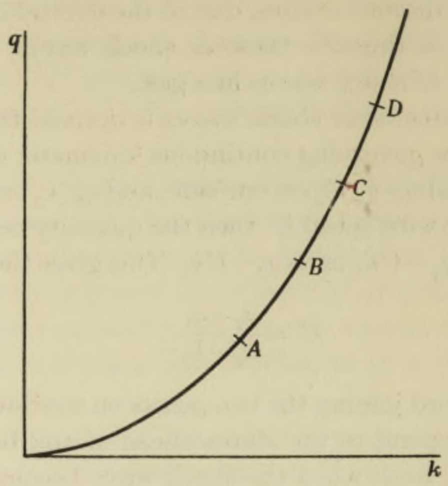

FIGURE 1

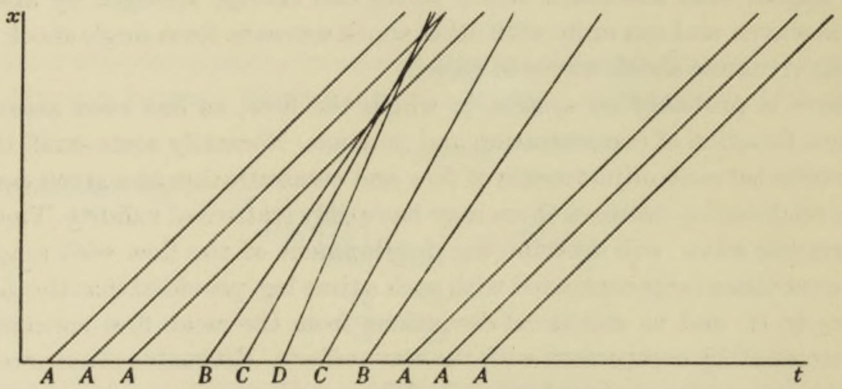

FIGURE 2

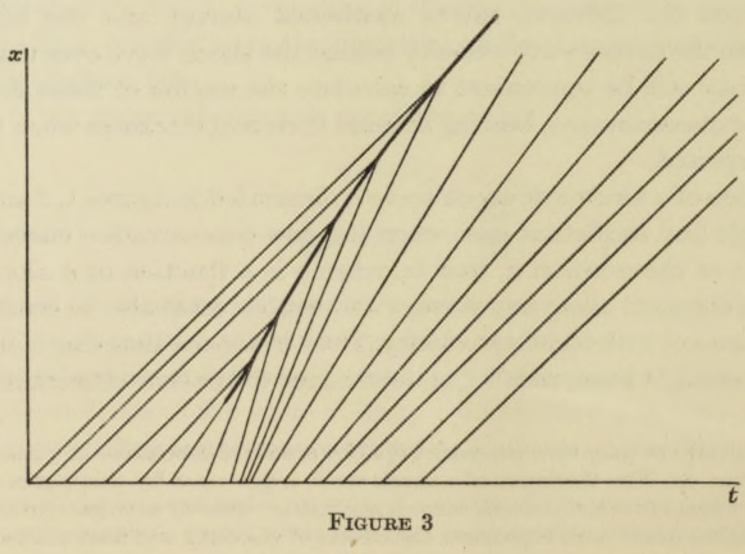

FIGURE 3

at the point which corresponds to the values of q and k carried by the wave; this makes their construction particularly easy.

When the flow-concentration curve changes with x, the waves are no longer straight lines. The path of the wave carrying a given flow q is still, however, predictable once for all. Thus, if we express the wave velocity c as a function of q and x, the path of the wave is

$$t = \int_0^x \frac{\mathrm{d}x}{c(q,x)} + \text{constant} = \int_0^x \left(\frac{\partial k}{\partial q}\right)_{x\ \text{constant}} \mathrm{d}x + \text{constant}. \tag{8}$$

When the integral in (8) has been calculated† for all values of q, the construction of the wave pattern presents hardly more difficulty than in the case illustrated in figure 2.

Figure 2 shows the progress of a 'hump', that is, a region of higher concentration in the midst of a region of uniform concentration. When the (q, k) curve is concave upwards, like that in figure 1, then the wave velocity $c = \mathrm{d}q/\mathrm{d}k$ increases with k, and hence increases also with x for the waves in the *rear* of such a hump. Accordingly, those waves spread out fanwise, getting ever farther apart. In the front of the hump, however, k and hence also c decrease with x, so that the waves there converge and finally cross. Obviously one cannot accept such a solution, in which the flow effectively has two values at some points. Fortunately, it is always possible to avoid this by fitting in a shock wave, as in figure 3 (which is drawn on a smaller scale than figure 2, to show the later development of the shock wave). Techniques for calculating the progress of the shock wave, from the condition that at each point its velocity is given by (7), in which q_1 and q_2 are the flows carried by two continuous waves which meet on the shock wave, are given below (§ 4). Figure 3 shows how the increase in concentration at the shock wave grows initially, and also how after a very long time the waves which meet on it are inclined to each other at a smaller angle again—that is, the increase is reduced. Thus the shock wave, and with it the hump, ultimately decay, as the shock wave passes farther and farther into the region of uniform concentration ahead of the hump.

If the curve of q against k is convex upwards (as in the problem of traffic flow discussed in part II), the wave velocity is *reduced* in the 'hump', and the shock wave appears in the rear. But its progress and decay are in other respects similar.

In some applications, including the case of flood waves (see below), kinematic waves and dynamic waves are both possible together. However, the dynamic waves have both a much higher wave velocity and also a rapid attenuation. Hence, although any disturbance sends some signal downstream at the ordinary wave velocity for long gravity waves, this signal is too weak to be noticed at any considerable distance downstream, and the main signal arrives in the form of a kinematic wave at a much slower velocity (§ 3).

Now, this situation is so parallel to the familiar behaviour of dynamic waves in a dispersive medium, where the energy of the vibrations in any narrow frequency

† This might be done most easily by calculating $\int_0^x k(q,x)\,\mathrm{d}x$ for different values of q, and differentiating with respect to q.

band travels not at the wave velocity but at the group velocity for that frequency band, that one is impelled to search very carefully for a way in which the behaviour of flood waves can be regarded as essentially a particular case of the group-velocity phenomenon. The search is fruitless, however, and the true relationship is different. The group-velocity phenomenon is itself essentially a particular case of the kinematic wave phenomenon, so that it is cognate to, rather than inclusive of, the behaviour of flood waves.

To understand the behaviour of a travelling wave in a dispersive medium from the point of view of kinematic wave theory, it is necessary only to choose the 'quantity' whose motion is discussed kinematically to be the *number of wave peaks*. Then the flow q (number of peaks passing per unit time) becomes the frequency, and the concentration k becomes the wave number. A functional relationship exists, of course, between them. The basic conclusion (4) of kinematic wave theory then states that the frequency q remains constant for points travelling with the velocity $\mathrm{d}q/\mathrm{d}k$; this is the group velocity if q and k have the meanings mentioned. This statement describes correctly the process of dispersion (compare the statement at the beginning of the last paragraph). If the kinematic waves spread out fanwise, that is, the energy in a narrow frequency band is spread over an increasingly larger region, then evidently the amplitude must vary as the inverse square root of the distance between successive kinematic waves (apart, that is, from any damping due to energy *dissipation* in that frequency band).

A difference from ordinary kinematic waves arises, however, when the wave groups cross, as in figure 2 above. There is no physical unreality about *this* crossing, since the frequency q can without difficulty take two values at the same place. For example, there are water waves of two frequencies (a capillary wave and a gravity wave) corresponding to any given value of the group velocity, and wave groups with these two frequencies can travel along together. Thus the modification of figure 2 into figure 3 does not happen in this application; no discontinuities in frequency can appear.

It is well known, of course, that group velocity can be explained kinematically. However, such simple physical explanations as have been given previously are unsatisfactory because the arguments apply only to a wave group with a small total frequency range. The above argument shows that the essential properties follow at once for an arbitrary wave train from applying 'conservation of number of wave peaks'. This conservation is not, of course, accurately true; but once sufficient dispersion has occurred to render small the frequency change in a single wavelength, then the appearance or disappearance of peaks through the occurrence of horizontal points of inflexion must become very rare. The additional argument bringing in energy shows that Kelvin's asymptotic formula for the travelling wave resulting from a limited initial disturbance can be deduced by elementary arguments in every respect except phase.† Further, the kinematic wave approach to the subject is usefully general; thus, it will show at once how the transmitted waves redisperse when an established train of waves enters another medium.

† Evidently, the $\mathrm{d}^2q/\mathrm{d}k^2$ in Kelvin's formula comes in as the rate of spread $\mathrm{d}c/\mathrm{d}k$ of kinematic waves of different velocities.

The reader will observe that kinematic wave theory is being advocated not only as an instrument for research, but also as a demonstrational method for deriving important results with a minimum of labour. It is in this connexion that we wish to point out that travelling dynamic waves of longitudinal type can be regarded as special cases of kinematic waves, and that this may give a conveniently simple way of deriving their non-linear properties. Thus for plane sound waves, if 'quantity' signifies mass per unit area perpendicular to the direction of flow, then the concentration k (quantity per unit length) becomes the ordinary density; the flow q becomes kv, where v is the fluid velocity. Now, for a wave travelling without energy dissipation in the direction x increasing, we have at all points Riemann's relation

$$v = \int_{k_0}^{k} a \frac{\mathrm{d}k}{k}, \tag{9}$$

where a is the velocity of sound for density k, and k_0 is the density of the undisturbed atmosphere. This gives a relationship between $q = vk$ and k, which corresponds to a kinematic wave velocity

$$c = \frac{\mathrm{d}q}{\mathrm{d}k} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} = v + a. \tag{10}$$

Note that the wave velocity a relative to the medium corresponds to a kinematic wave velocity (always a velocity in *space*) $v+a$. It will be seen from (10) that Riemann's relation is inevitable kinematically if waves are to exist which travel unchanged with velocity a relative to the medium.

The theory of the formation of shock waves in a gas is then a special case of the general theory given above. It should be remarked that it will be only approximate, as equation (9) (the constancy of Riemann's invariant) is not accurately true in the region behind a shock wave. However, that equation is a very good approximation for shock waves of moderate strength, and is normally used in all attempts to calculate shock wave movement.

What has been said applies equally to the behaviour of long gravity waves in a channel of constant width and horizontal bottom. If 'quantity' signifies volume of water per unit width, the concentration k becomes the local depth of the water; equations (9) and (10) are correct with $a = \sqrt{(gk)}$, and the kinematic shock wave is now a bore.

But, however convenient such devices may be for developing the theory of a number of important phenomena from a simple and unified point of view, one must not forget that in these last two problems the system is only a kinematic wave system if attention be deliberately restricted to waves travelling in one direction only. The methods cannot be used to treat reflexion (after which Riemann's relation (9) ceases to be valid), and, indeed, in a true kinematic wave system no reflexion of any kind is possible (mathematically, there is only one system of characteristics).

The rest of this paper, and part II, are devoted to kinematic wave systems which are more 'robust' in that the hypotheses remain essentially valid under a wider range of states of the system.

2. Flood waves

Although we believe that the full kinematic wave theory as set out in § 1, comprising the theory of continuous waves, 'shock waves' and the formation of shock waves out of continuous waves, has not previously been given, parts of it have been known (though never widely known) for almost a century in their application to flood movement in rivers.

In particular, several writers independently have given the theory of continuous kinematic waves, based on equations (1) to (4), as applying to flood movement. Boussinesq (1877) gives a full treatment, including a derivation of equation (8), as does Forchheimer (1930) in his invaluable book *Hydraulik*. They refer to Kleitz (1858, unpublished), Breton (1867) and Graëff (1875) as pioneers of the theory.

The earliest account in the English language is by Seddon (1900), who discusses the problem at length with special reference to the Mississippi and its tributaries. Seddon was unaware of the earlier work. In some ways, however, his account (which, conversely, was unknown to Forchheimer (1930) and has received only rather perfunctory reference even in the later American literature) is to be preferred. This is because he shows a greater understanding of the variety of mechanisms which govern the relationship between flow and concentration, and such understanding is vital for sound application or improvement of the theory.

In the best known of these mechanisms, a balance is struck between the friction of the bottom and the component of gravity in a direction which is downstream and parallel to the free surface of the river.† If the downward slope of the free surface is S, then the gravitational force per unit length of river is $\rho g S k$, where k is the concentration (volume of water per unit length). The frictional force per unit length may be expressed as $f \rho v^2 L$, where v is the mean velocity, L the wetted perimeter of the cross-section, and f a coefficient of friction. Equating these forces, we get

$$v = \sqrt{\frac{gSk}{fL}}. \tag{11}$$

In the alternative form $v = C\sqrt{(RS)}$, where $R = k/L$ (area of cross-section divided by wetted perimeter, or 'hydraulic mean depth') and $C = \sqrt{(g/f)}$, this is the famous 'Chézy formula'. It shows that $v \propto k^{\frac{1}{2}}$, and hence that the wave velocity, by (6), is

$$c = v + k\frac{\mathrm{d}v}{\mathrm{d}k} = \tfrac{3}{2}v, \tag{12}$$

provided that none of S, f or L vary with k.

In practice at least one of these shows some variation with depth. Thus the friction coefficient f depends on the ratio of the size of typical roughness elements in the bed to the hydraulic mean depth R. This relationship, as predicted by turbulence theory and borne out by the best experiments, is logarithmic, but an approximation

† This direction is appropriate because the force due to hydrostatic pressure gradient has zero component along it, and so need not be considered.

to it, reasonably accurate in typical conditions of river flow, is Manning's relation $f \propto R^{-\frac{1}{3}}$. For constant S and L this gives

$$v \propto k^{\frac{2}{3}}, \quad c = \tfrac{5}{3}v. \tag{13}$$

Again, the wetted perimeter L increases significantly with k for many shapes of river cross-section. What may seem a fairly extreme case is a triangular cross-section (apex downwards), for which $L \propto k^{\frac{1}{2}}$. This reduces c from $\frac{3}{2}v$ (equation (12)) to $\frac{5}{4}v$ if f is a constant, and from $\frac{5}{3}v$ (equation (13)) to $\frac{4}{3}v$ if $f \propto R^{-\frac{1}{3}}$. Even greater reductions are possible however; thus, in many reaches of the Mississippi and its tributaries, even the *ratio* of width to depth increases with depth, due to the erosion of narrow channels with convex sides in the river bed; in such extreme cases, c may exceed v hardly at all.

The French and German writers recognized only the mechanisms cited above, and regarded S as independent of k, equating it (as far as the work on kinematic wave theory is concerned) to the mean slope of the bed. It is evident, however, that even when the flow q is uniform along the river, the slope S of the free surface will differ from that of the bed wherever the cross-section is changing with x; where the river is widening, for example, S must exceed the mean slope of the bed. Under these circumstances, S might vary with k for fixed x, which would affect the value of c.

Seddon (1900) made a more fundamental criticism of any approach to river flow which is based solely on the Chézy formula and extensions to it. Put simply, his objection is that great rivers, unlike man-made conduits, do not have a uniformly sloping bed, nor do they in any way approximate to this condition. First, the slope of the bed exhibits enormous variations (including changes of sign) in the *small*, that is, across the width of the river and over distances downstream comparable with the width. Even more seriously, the large-scale configuration of the bed is frequently very much like a series of 'pools and bars'. At relatively low water the flow from one pool to the next is then determined not so much by a velocity-slope relationship but by the relations governing the flow over a 'submerged weir'—that is, over one of the bars. Large values of the slope S of the free surface are confined to the neighbourhood of these bars. The same stretch of river might, however, be governed by quite a different mechanism at high water. Then, for example, parts of the 'pools' might become by far the narrowest sections of the river, and control the flow like an orifice with vertical walls.

Another point observed by Seddon, where alluvial rivers are concerned, is that the bed is constantly changing with time, since its material is readily handled by the flow. Variations in depth of 10 ft. about its mean at a point are common on the Lower Mississippi; at the same time the elevation of the surface changed only by an inch or two for the same value of the flow q. Thus the height of the free surface above some fixed horizontal plane varies far more smoothly in time, as well as in space, than the depth of the bottom.

Seddon used the symbol h, and the word 'stage', to denote, at each point on the river, the height of the free surface above a certain reference plane, fixed as far as that point is concerned. It is convenient to regard the flow q as primarily a function of the stage h rather than the area k of the water cross-section—both because such a relation has more permanence, as we have just seen, and because h is more easily measurable.† For constant x,

$$\mathrm{d}k = B\,\mathrm{d}h, \tag{14}$$

where B is the local breadth of the river.

Hence (3) becomes

$$c = \frac{1}{B}\left(\frac{\partial q}{\partial h}\right)_{x\ \text{constant}}, \tag{15}$$

and this form for the wave velocity is often far less susceptible to variation with time, or dependence on the taking of averages, than the standard definition (3). The two can be reconciled, however, if in § 1 'quantity' is taken to mean 'volume of water above the low-water mark'.

Seddon concludes from his long and interesting physical discussion, of which just the salient points have been mentioned above, that the factors which go to make up the relationship

$$q = q(h, x), \tag{16}$$

between flow and stage at different stations on the river, are nearly always too complicated to make the *prediction* of this relation a sensible direction in which to apply scientific method. Rather, this static relationship should be determined by observation, when, in spite of the endless variety and complication of the processes involved, it is nevertheless found to have some permanence and reliability. It may then be used, with a knowledge of the breadth

$$B = B(h, x) \tag{17}$$

as a function of stage and position, to predict from equation (15) for the wave velocity c the still more complicated, dynamic, phenomena involved in flood movement. Conversely, Seddon has so much confidence in this relation (15) that he would regard the measured speeds of propagation c of particular values of the flow q down the river, together with *one* of the relations (16) and (17), as a reasonable method of obtaining the other relation! To sum up, equation (15) is the one basic law to which a river will conform.

We have stated Seddon's views in their original, somewhat exaggerated, form to draw attention to the danger of concentrating on velocity-slope relations when dealing with rivers, as opposed to man-made conduits. Our own view is not that such relations are valueless in all cases, but that a general theory should avoid leaning heavily on them. Again, we do not claim that the kinematic wave theory gives a really exact model of flood movement. The literature already contains methods of improving the approximation. Thus Forchheimer (1907; 1930, p. 299) gives an expression for the rate of subsidence of the peak of a flood wave, obtained by applying the Chézy formula without neglecting the contribution of stage gradient $(-\partial h/\partial x)$ to the slope S of the free surface. Thomas (1934, 1940) has devised step-by-step methods of 'flood routing' based on equations of motion which take this effect into account together with the (smaller) effect of inertia. Lin (1947) treats the same equations by the numerical method of characteristics. The characteristics

† He suggests that the reference height $h = 0$ may be taken as that corresponding to a particular constant flow q_0, the lowest observed on the river; thus $q = q_0$ for all x when $h = 0$.

here are the paths of the dynamic waves associated with the problem, namely, long gravity waves.

In §§ 4 and 5 we give a new procedure for predicting flood movement, which is bound up more with the kinematic wave as a first approximation, and less with velocity-slope relations than the methods cited. First, however, in § 3, we have thought it desirable to give a mathematical treatment of the 'competition' between kinematic and dynamic waves in river flow, in order to show how completely the dynamic waves are subordinated in the case of greatest interest, that is, when the speed of the river is well subcritical. This demonstrates the unsuitability of the characteristics of the dynamic wave system as a basis for computation. In § 3 we show also how the situation is different in supercritical streams, in which the kinematic and dynamic waves can play equally important parts. The 'roll waves' observed in mountain streams, as analyzed by Dressler (1949), are a case of this. Readers interested only in procedures for flood prediction are advised to omit § 3 at a first reading.

The process by which kinematic waves may steepen into 'shock waves', with a considerable change in flow occurring in a relatively short distance, has not been very clearly expressed in the flood-wave literature. However, the possibility of such a wave progressing down the river, with different, uniform, flows upstream and downstream of it, has been envisaged as a model of a flood, and its difference from a bore (the 'dynamic' analogue) clearly seen. Such a wave has been called a 'monoclinal flood wave', or 'steady profile'. The formula (7) for its velocity is given by Boussinesq (1877, p. 479). Calculations of the shape of the profile, from the full friction-slope-inertia equations of motion, have been made by Thomas (1937).† (See also § 3 below, and Dressler (1949), who uses them in his theory of roll waves.) The length of the monoclinal flood wave (or 'shock wave thickness') is found to be of the order of magnitude h/S, which is the distance downstream in which the river elevation falls by an amount equal to its depth.

However, it will be seen from the discussion in § 1, and in particular from figure 3, that while a kinematic shock wave may play a very important part in the forward regions of a flood wave, it does not constitute the whole wave. In particular, it cannot correctly be regarded as remaining uniform in strength, or as having uniform conditions both upstream and downstream of it. In fact, its growth and decay, due to interaction with continuous waves on both sides, are an essential part of the flood-wave phenomenon. The new way of using the kinematic shock wave, which figure 3 illustrates, is correspondingly an essential part of the method of predicting flood movement described below in §§ 4 and 5.

3. Competition between kinematic and dynamic waves

The object of this section is to bring out the mathematical relations between kinematic and dynamic waves, and to demonstrate their relative importance under various flow conditions in which both are present. For this purpose it is sufficient

† The authors have been unable to consult Thomas's papers, which are very inaccessible. Accordingly, no reference to their details can be made in § 3 where the solutions are discussed.

to consider one only of the many possible mechanisms governing the propagation of kinematic waves which were described in § 2. In fact, we choose the most straightforward of these, namely, the balance between slope and friction, as expressed in equation (11). However, in order that dynamic waves can be present, we can no longer neglect the inertia of the fluid, or the dependence of the slope S of the free surface on the gradient along the river of the stage h. Accordingly the difference of the gravitational and frictional forces per unit mass of fluid (from § 2, this difference is $(\rho g S k - \rho v^2 f L)/\rho k$) is set equal to the acceleration of the fluid, to give

$$v_t + v v_x = g\left(S - \frac{v^2}{C^2 R}\right), \tag{18}$$

in place of (11). If the stage h is introduced, we have $S = S_0 - h_x$, where S_0 is the value of S for uniform stage; hence, since C and R are functions of h, (18) provides a differential relation between h and v. A second equation for these two quantities is provided by the continuity equation (1), since q and k can be expressed in terms of h and v.

There seems little doubt that the general characteristics of the competition between kinematic and dynamic waves will be reproduced clearly in any formulation of the problem which still retains all the essential features. (Practical methods of prediction are postponed to § 4.) Accordingly, in this section we make the following additional simplifications:

(i) It is assumed that by taking the reference surface for the stage as the average position of the river bed, the hydraulic mean depth R may be approximated by the stage h.

(ii) The slope S_0 of the reference surface is taken to be constant.

(iii) The Chézy resistance law is used, i.e. C is constant.

(iv) In the undisturbed flow, we suppose that h and v take constant values, h_0 and v_0, say.

Then, since $q = kv$ and $k = Bh$ where the breadth B is constant, the equation of continuity is

$$h_t + v h_x + h v_x = 0. \tag{19}$$

Equation (18) becomes

$$v_t + v v_x = g\left(S_0 - h_x - \frac{v^2}{C^2 h}\right), \tag{20}$$

and in the undisturbed flow the values of stage and velocity are related by

$$v_0 = C\sqrt{(h_0 S_0)}. \tag{21}$$

Equations (19) and (20) are often assumed in the literature when theoretical aspects of river flow are being considered. They follow immediately when the river is idealized as a uniform rectangular channel with slope S_0 and sufficiently wide for the hydraulic mean depth to be approximated by the depth h, but the above assumptions are rather less severe.

Kinematic waves are obtained by neglecting the derivative terms in (20). Then $v \propto h^{\frac{1}{2}}$, and from (19),

$$h_t + \tfrac{3}{2} v h_x = 0, \tag{22}$$

showing that h and v remain constant for waves travelling downstream with velocity $\frac{3}{2}v$. On the other hand, without the terms gS_0 and v^2/C^2h, (19) and (20) are the equations of the usual theory of long gravity waves. In that case it is well known (see, for example, Stoker 1948) that the solution represents systems of waves moving upstream and downstream, both with speed $\sqrt{(gh)}$ relative to the flow. In our terminology, these waves are *dynamic* and the turbulent bores, which occur if waves break, are dynamic shocks. Mathematically, the wave property is recognized from the characteristics of the equations; since (19) and (20) are equivalent to a second-order equation, they have two systems of characteristic curves, given by $dx/dt = v+\sqrt{(gh)}$ and $dx/dt = v-\sqrt{(gh)}$ respectively. Moreover, inclusion of the additional terms in (20) does not change the characteristics since they are determined by the derivative terms alone. Hence, dynamic waves always occur. The additional friction and slope terms can only modify the amplitude of these waves. Under the conditions appropriate for flood waves, however, they do this to such a degree that the dynamic waves rapidly become negligible, and it is the kinematic waves, following at a slower speed, which assume the dominant role.

The decay of the dynamic waves can be demonstrated very simply. Discontinuities in derivatives of v and h may be taken as typical disturbances. They will propagate upstream and downstream with the appropriate characteristic velocities (this is in fact a defining property of characteristics) and the variation in the magnitudes of the discontinuities can be specified immediately from (19) and (20). The results give the rate of growth or attenuation of the disturbances carried by the dynamic waves. The standard procedure is described in Courant & Hilbert (1937, p. 359) and the results for the present problem have been noted by Masse (1938). Avoiding reference to the general theory of characteristics, one may simply expand h and v in power series near the 'wave-front'. (The wave-front is the first disturbance and propagates with the characteristic velocity appropriate to the undisturbed flow.) For downstream propagation, the wave-front is $\tau = 0$, where $\tau \equiv t-x/(v_0+\sqrt{(gh_0)})$, and the expansions are

$$v = v_0+\tau v_1(t)+\tau^2 v_2(t)+\dots,$$
$$h = h_0+\tau h_1(t)+\tau^2 h_2(t)+\dots.$$

Here, the first derivatives of v and h are discontinuous, but the argument goes through with the same result if higher-order derivatives are the first discontinuous ones. The discontinuities in $\partial h/\partial t$ and $\partial h/\partial x$ at the wave-front are $h_1(t)$ and $-h_1(t)/(v_0+\sqrt{(gh_0)})$, respectively; hence the growth or decay in their magnitudes as the wave-front travels downstream are determined by $h_1(t)$. Substituting the expansions in (19) and (20), and replacing C from (21), it is found that

$$\frac{dh_1}{dt} = \frac{3}{2h_0(1+F)}h_1^2 - \frac{gS_0}{v_0}(1-\tfrac{1}{2}F)\,h_1, \tag{23}$$

where F is the Froude number $v_0/\sqrt{(gh_0)}$ (F plays a role analogous to the Mach number in gas flow). We are interested in the case in which h_1 is initially positive. Then, if $F>2$, it is clear from (23) that h_1 increases without limit; since h_1 is proportional to $\partial h/\partial x$, this means that the face of the wave becomes vertical and the wave breaks into a bore. If $F<2$, the sign of the right-hand side of (23) depends on whether h_1 is initially greater or less than

$$K = \left(\frac{gh_0S_0}{3v_0}\right)(2-F)(1+F). \tag{24}$$

If $h_1(0) > K$, h_1 again increases indicating bore formation; if $h_1(0) < K$, h_1 tends to zero. The last case is the relevant one for flood waves; we even assume that the flow is subcritical ($F < 1$), and values of $\partial h/\partial t$ as large as K are never found. Under these conditions, the solution of (23) is

$$h_1(t) = \frac{Kh_1(0)\,\mathrm{e}^{-bt}}{K - h_1(0)\,(1-\mathrm{e}^{-bt})}, \tag{25}$$

where $b = gS_0(1-\frac{1}{2}F)/v_0$; *thus the decay is exponential.* The dynamic waves are rapidly damped out, and bore formation is prevented.

The formation of a bore in other cases does not in itself imply that the dynamic waves are any more important; the strength of the bore may decrease just as rapidly. We shall indicate later that this is so when F is appreciably less than 2; but when $F > 2$, an approximate theory predicts that the strength increases without bound and the theory ceases to apply.

The criterion of $F \gtrless 2$, or from (21) its equivalent form $S_0C^2/g \gtrless 4$, is satisfactory since other considerations show that $F = 2$ will be critical. If $\frac{3}{2}v_0$ and $v_0 + \surd(gh_0)$ are taken as typical velocities for kinematic and dynamic waves, respectively, $F = 2$ is the value at which these velocities become equal. If $F = 2$, the energy carried by the kinematic waves goes along with the dynamic wave front; if $F > 2$, kinematic waves cannot carry the energy (continuously), and the general description of flood waves given earlier would cease to apply. Again, Dressler (1949), Dressler & Pohle (1953) found that $S_0C^2/g > 4$ is a necessary condition for the instability of steady flow and the formation of roll waves. This fits in admirably with the above results.

The critical value of F for resistance laws other than the Chézy law may be deduced most simply by equating the kinematic and dynamic wave velocities. Thus the critical value is when $\surd(gh) = h\,\mathrm{d}v/\mathrm{d}h$ where $v = C\surd(S_0h)$. When $v \propto h^n$, for example, this gives $v/\surd(gh) = 1/n$; for the Manning formula $n = \frac{2}{3}$, leading to $F = \frac{3}{2}$.

We postpone further discussion of the more extreme flows and return to the question of the roles of kinematic and dynamic waves in flows with $F < 1$. This question has now been elucidated to some extent by separate consideration of the two types of wave; the discussion will be completed by an account of the linear theory of small disturbances. The approximations of linearization are, in some respects, severe, but the compensating advantage of the theory is that a complete solution *containing both kinematic and dynamic waves* can be found. The main effects of non-linearity may be sketched in afterwards. The linear equations are obtained by substituting $v = v_0 + u$, $h = h_0 + \eta$ in (19) and (20), and retaining only first-order terms in u and η. Thus, using (21), the equations may be approximated by

$$u_t + v_0u_x + g\eta_x + gS_0\left(\frac{2u}{v_0} - \frac{\eta}{h_0}\right) = 0, \tag{26}$$

$$\eta_t + v_0\eta_x + h_0u_x = 0. \tag{27}$$

A single equation for η is obtained by differentiating (26) with respect to x and substituting for u_x from (27), to give

$$(gh_0 - v^2)\,\eta_{xx} - 2v_0\eta_{xt} - \eta_{tt} - 2\lambda\eta_t - 3\lambda v_0\eta_x = 0, \tag{28}$$

where $\lambda = gS_0/v_0$. Making a rather surprising appearance in this subject, equation (28) is a form of the 'telegraph equation' which occurs in the study of transmission lines; methods for its solution are, therefore, well known.

The typical initial-value problem for flood waves is as follows. For $t \leqslant 0$, the river downstream of a certain observation point, $x = 0$ say, is undisturbed with $h = h_0$, $v = v_0$. Starting at $t = 0$, a flood wave passes the point $x = 0$, and one quantity, h say, is observed as a function of t. Hence, the boundary conditions are that for $t = 0$, $\eta = \eta_t = 0$ in $x > 0$, and η is equal to a given function $f(t)$ at $x = 0$; the value of η is required in $x > 0$ and $t > 0$. The problem is well set provided $F < 1$ and the derivation of the solution is standard; both Riemann's method and the operational method have been applied to its solution (Deymie 1938; Masse 1938). Here we employ the Heaviside calculus† in which the operation $\int_0^t dt$ is denoted by $1/p$. Then, if $Y(p, x)$ is the representation of η, equation (28) becomes

$$(gh_0 - v_0^2)\,Y_{xx} - (2p + 3\lambda)\,v_0 Y_x - (p^2 + 2\lambda p)\,Y = 0.$$

The general solution is

$$Y = A_1(p)\,e^{P_1(p)x/\sqrt{(gh_0)}} + A_2(p)\,e^{P_2(p)x/\sqrt{(gh_0)}}, \tag{29}$$

where P_1 and P_2 are the roots of the quadratic

$$(1 - F^2)\,P^2 - (2p + 3\lambda)\,FP - (p^2 + 2\lambda p) = 0,$$

and A_1, A_2 are arbitrary functions of p; solving the quadratic, we have

$$P_1, P_2 = \frac{(p + \frac{3}{2}\lambda)\,F \pm \sqrt{(p^2 + \lambda p(2 + F^2) + \frac{9}{4}\lambda^2 F^2)}}{1 - F^2}. \tag{30}$$

For large p, the behaviour of the roots is given by

$$P_1 \sim p/(1 - F), \quad P_2 \sim -p/(1 + F);$$

hence, the first term in (29) represents waves travelling upstream and is zero for $x < -t\{\sqrt{(gh_0)} - v_0\}$, while the second term represents waves travelling downstream and is zero for $x > t\{\sqrt{(gh_0)} + v_0\}$. For our problem, therefore, only the second term can appear, and we drop the suffix 2, taking

$$Y(p, x) = A(p)\,e^{P(p)x/\sqrt{(gh_0)}}, \tag{31}$$

where P is (30) with the negative sign for the square root. Since $\eta = f(t)$ for $x = 0$, it follows immediately that $A(p)$ must be the operational representation of $f(t)$. The interpretation of (31) as an integral involving a Bessel function may be obtained by application of the usual methods of Heaviside calculus, but the expressions are lengthy and the details are relegated to the Appendix; the results required here can be deduced directly from (31).

We are interested in two questions. First, the behaviour of the solution near the dynamic wave-front, and, secondly, the location of the main disturbance. The

† Readers more familiar with the Laplace transform may note that the Heaviside representation of a function differs from the Laplace transform by an additional factor p.

values of η near the wave-front correspond to the values of $Y(p,x)$ for large p. For large p,

$$Y = A(p)\exp\left\{-\frac{px}{v_0+\surd(gh_0)}-\frac{\lambda x}{\surd(gh_0)}\frac{1-\frac{1}{2}F}{1+F}+O\left(\frac{1}{p}\right)\right\},$$

and the interpretation of this is

$$\eta = \exp\left\{-\frac{\lambda x}{\surd(gh_0)}\frac{1-\frac{1}{2}F}{1+F}\right\}f\left(t-\frac{x}{v_0+\surd(gh_0)}\right)\left\{1+O\left(t-\frac{x}{v_0+\surd(gh_0)}\right)\right\}. \tag{32}$$

This expression describes the dynamic waves and their exponential decay is confirmed. In fact, the linear approximation of (25) is $h_1 \propto \exp\{-\frac{1}{4}\lambda(2-F)\,t\}$ which is in exact agreement with (32) on the wave-front.

Although the solution (31) requires $F<1$ (otherwise the probem is not correctly set since the other family of characteristics $t+x/\{\surd(gh_0)-v_0\}$ = constant also carries disturbances downstream), η is given near the wave-front by (32) in all cases; hence, the prediction that for $F>2$ the disturbance increases is also valid. Since we are using a linear theory, the propagation speeds, $v+\surd(gh)$, of the individual wavelets are approximated by the same value $v_0+\surd(gh_0)$ and the possibility of bore formation due to later wavelets overtaking the wave-front is excluded. This failing may be corrected, however, by methods already used in analogous problems of gas dynamics to predict and determine shocks from an improved linear theory (Whitham 1952). It is then found that a bore will ultimately be formed provided that $f'(t)$ exceeds the value K (equation (24)) for some range of t. (This is a valuable check with the earlier result.) The strength of the bore can be evaluated, and when $F<2$, it is found that after an initial formation period, the strength decays exponentially at the rate shown by (32). When $F>2$, the whole disturbance, including the strength of the bore, is predicted to become large and this theory breaks down. It must be remarked that the result for $F<2$ has been deduced under the assumption that the disturbance is small. More generally, as will be seen below, the value of F which must be exceeded if a bore of constant strength is to be maintained, depends on the strength; this value of F is always less than 2 and tends to 2 as the strength approaches zero. The required conditions for bore formation and the prediction of bore height are of more interest for the bores which are formed in certain rivers by the rising tide. This problem has been investigated by M. Abbott, and his results will be published in due course. In that case the bore decays under all conditions, at least in a uniform channel, a result which we associate with the fact that kinematic waves do not propagate upstream; a permanent increase in the height at the mouth of the river is accommodated through a *steady* flow in the river (the profile of its surface being a so-called 'backwater curve'), and the bulk of the disturbance does not propagate upstream at all.

In order to show the main features of the solution away from the wave-front and, in particular, to locate the main disturbance, an approximate form of the solution for large values of t is next obtained. To be precise the solution for large t is found in the range $t\delta < x/\{v_0+\surd(gh_0)\} < (1-\delta)\,t$, where δ is any small positive number. The reason for this condition will become apparent later; it excludes only the initial observation point and the wave-front where the flow is already known.

We consider two problems: (*a*) the original disturbance is a 'hump' with $f(t)$ returning to zero after a sufficient time, and (*b*) the disturbance is a 'smoothed step' so that $f(t)$ tends to a constant positive value at $t \to \infty$. Problem (*b*) follows from the solution to (*a*), since η_t satisfies exactly the same conditions in (*b*) as does η in (*a*). Therefore, we consider first the case when $f(t) \to 0$ as $t \to \infty$, and we shall require that $\int_0^\infty f(t)\,\mathrm{d}t$ is convergent.

The solution η may be expressed in terms of $Y(p,x)$ by the contour integral

$$\eta(x,t) = \frac{1}{2\pi \mathrm{i}}\int_{l-\mathrm{i}\infty}^{l+\mathrm{i}\infty} \frac{Y(p,x)\,\mathrm{e}^{pt}}{p}\,\mathrm{d}p,$$

where l is so large that all the singularities of $Y(p,x)$ lie to the left of the path of integration. Introducing the expression for $Y(p,x)$, and for convenience setting $m = x/t\sqrt{(gh_0)}$, we have

$$\eta = \frac{1}{2\pi \mathrm{i}}\int_{l-\mathrm{i}\infty}^{l+\mathrm{i}\infty} \frac{A(p)}{p}\,\mathrm{e}^{t(mP+p)}\,\mathrm{d}p.$$

The behaviour of η for large t is now found by estimating the integral by the method of steepest descents. The asymptotic expansion (of which we shall find the first term) will be valid for values of t which are large compared to some quantity having the dimensions of time and, in fact, the precise condition† is $t \gg 1/\lambda$. The contour is chosen to pass through the saddle-point of the function $mP(p)+p$ which is where

$$1 + mP'(p) = 0; \tag{33}$$

the main contribution to the integral then comes from the neighbourhood of the saddle-point. For large t, we have, according to the standard formula for this method,

$$\eta \sim \frac{1}{\sqrt{\{2\pi\,|P''(p_0)|\,tm\}}}\,\frac{A(p_0)}{p_0}\,\mathrm{e}^{\{p_0+mP(p_0)\}t}, \tag{34}$$

where $p = p_0(m)$ is the solution of (33). (This formula does not apply at $x = 0$, or at the wave-front where $m = 1+F$, since $P''(p_0)$ vanishes for this value of m.) For fixed t, the exponential term (which dominates the expression) is maximum for the value of m which is given by

$$\frac{\mathrm{d}}{\mathrm{d}m}(p_0 + mP(p_0)) = 0.$$

But in view of (33) this reduces to $P(p_0) = 0$. The zero of $P(p)$ is $p = 0$; hence η attains its maximum when $m = -1/P'(0)$. From (30) we deduce that $m = \frac{3}{2}F$, i.e. $x = \frac{3}{2}v_0 t$. *Hence the position of maximum depth ultimately travels downstream with velocity $\frac{3}{2}v_0$, showing that the main disturbance is carried downstream as a kinematic wave.* To find the value of the maximum depth we substitute $p_0 = 0$ in (34), noting that $A(p_0)/p_0$ is now replaced by $\lim_{p\to 0} A(p)/p = \int_0^\infty f(t)\,\mathrm{d}t$. Thus we find that

$$\eta_{\max.} \sim \left(\frac{\lambda}{2\pi t}\right)^{\frac{1}{2}} \frac{3F}{2(1-\frac{1}{4}F^2)^{\frac{1}{2}}}\int_0^\infty f(t)\,\mathrm{d}t. \tag{35}$$

† The argument $t(mP+p)$ may be written in dimensionless form as $\lambda t\{mQ(q)+q\}$, where $q = p/\lambda$ and $Q = P/\lambda$.

For problem (*b*) it is the position of maximum slope which moves downstream with speed $\frac{3}{2}v_0$, and the magnitude of the maximum slope falls off as $1/\sqrt{t}$.

In interpreting the results, it is essential to remember that the linear theory includes the 'diffusion' effects but does not include the equally important non-linear features. The latter would introduce modifications in the same way as for dynamic waves. That is, the kinematic wavelets instead of being lumped together with the same propagation speed should have individual speeds $\frac{3}{2}v$, taking into account the variation in v; in regions which have higher values of v the wavelets travel with higher velocity. In particular, in problem (*b*) the diffusion which acts to smooth out the step (the slope decreasing as $1/\sqrt{t}$) is counteracted by the non-linear steepening due to the higher values of v in the rear. The two opposing effects eventually achieve a balance and the wave is translated down the river without change in shape. This 'steady profile wave' is nothing but the kinematic shock separating constant flow regimes.

In (*a*), the propagation speed will be greatest at the peak, producing tendencies to steepen near the front and smooth out at the rear. Near the front, equilibrium between diffusion and non-linear steepening will be attained as in (*b*), and a shock appears at the head. The flow will be as represented in figures 2 and 3. The detailed solution of this problem is worked out in § 4, and it is found that the strength of the shock decreases like $1/\sqrt{t}$ for large t. Thus, for this case non-linearity distorts the profile and concentrates the disturbance near the head; the strength remains proportional to $1/\sqrt{t}$ as in the linear theory but the constant of proportionality is different.

A necessary condition for the approximation (35) for large t was that $t \gg 1/\lambda$. Therefore $1/\lambda$ provides an estimate of the time-scale which is required if the theory of kinematic waves is to be applied. This may also be noted directly from (28); for, if $t \gg 1/\lambda$, one is led to approximate the equation as

$$(gh_0 - v_0^2)\,\eta_{xx} - 2\lambda(\eta_t + \tfrac{3}{2}v_0\eta_x) = 0. \tag{36}$$

Without the diffusion term, η_{xx}, the equation has solution $\eta = \eta(x - \frac{3}{2}v_0 t)$, representing kinematic waves. Therefore, the full equation (36) represents waves travelling with speed $\frac{3}{2}v_0$, but with amplitude decreasing like $1/\sqrt{t}$ (as is typical in diffusion problems). Equation (36) also indicates that when appreciable changes in η_x occur only over distances x which are large compared to $gh_0/\lambda v_0$, the diffusion may be neglected, and the solution $\eta = \eta(x - \frac{3}{2}v_0 t)$ taken. Since $gh_0/\lambda v_0 = h_0/S_0$, this confirms the earlier remarks that appreciable diffusion is limited to relatively thin shock waves whose thickness (as we shall also see below), is of order h_0/S_0, and the problem may be treated accordingly.

The steady profile solution or 'monoclinal flood wave', which the foregoing arguments indicate as the ultimate wave-form in problem (*b*), is of great importance in the subject. It may be determined exactly by assuming in (19) and (20) that h and v are functions of a single variable $\sigma = x - Ut$; this is equivalent to describing the wave relative to axes moving with the velocity U of the wave, in which the flow is steady. The equation of continuity integrates to the obvious form for steady flow relative to the moving axes

$$h(U - v) = Q, \tag{37}$$

where Q is a constant. The values of U and Q are determined if, for example, the limiting values of depth at large distances ahead of and behind the wave are specified. If these values are h_0 and h_1, respectively, the corresponding values of v are $v_0 = C\sqrt{(S_0 h_0)}$, $v_1 = C\sqrt{(S_0 h_1)}$, since at large distances the flow tends to be uniform. Then it follows from (37) that

$$U = \frac{v_1 h_1 - v_0 h_0}{h_1 - h_0} = CS_0^{\frac{1}{2}} \frac{h_1^{\frac{3}{2}} - h_0^{\frac{3}{2}}}{h_1 - h_0}, \tag{38}$$

and

$$Q = h_0 h_1 \frac{v_1 - v_0}{h_1 - h_0} = CS_0^{\frac{1}{2}} h_0 h_1 \frac{h_1^{\frac{1}{2}} - h_0^{\frac{1}{2}}}{h_1 - h_0}. \tag{39}$$

The expression for U is the simplified form (appropriate for the simplification of this section) of (7). It reduces, for small values of $(h_1 - h_0)/h_0$, to $\frac{3}{2}v_0$ in accordance with linear theory. More accurately (and this will be referred to later), U is given by

$$\tfrac{1}{2}(\tfrac{3}{2}v_1 + \tfrac{3}{2}v_0) + O(v_1 - v_0)^2, \tag{40}$$

i.e. the mean value of the propagation speeds of the kinematic wavelets on each side of the wave.

The shape of the profile is given by the momentum equation (20) as the solution of

$$(v - U)\frac{\mathrm{d}v}{\mathrm{d}\sigma} = g\left(S_0 - \frac{\mathrm{d}h}{\mathrm{d}\sigma} - \frac{v^2}{C^2 h}\right),$$

which, from (37), may be written as

$$\frac{\mathrm{d}h}{\mathrm{d}\sigma} = -\frac{(Q - Uh)^2/C^2 - S_0 h^3}{h^3 - Q^2/g}. \tag{41}$$

Substituting for the values of Q and U in terms of h_0 and h_1, it becomes

$$\frac{\mathrm{d}h}{\mathrm{d}\sigma} = -S_0 \frac{(h - h_0)(h_1 - h)(h - H)}{h^3 - h_c^3}, \tag{42}$$

where

$$H = \frac{h_0 h_1}{(h_0^{\frac{1}{2}} + h_1^{\frac{1}{2}})^2} < h_0, \quad h_c^3 = Q^2/g; \tag{43}$$

here h_c is the critical depth for the flow relative to the moving system of reference. With $h_1 > h > h_0$ the sign of $\mathrm{d}h/\mathrm{d}\sigma$ depends on the sign of $h - h_c$; for flood waves, however, we may take $h > h_c$ and $\mathrm{d}h/\mathrm{d}\sigma$ is negative, giving the monoclinal wave. To integrate (42) we write

$$\frac{S_0}{h_0}\frac{\mathrm{d}\sigma}{\mathrm{d}h} = \frac{1}{h_0} - \frac{a_0}{h - h_0} - \frac{a_1}{h_1 - h} - \frac{A}{h - H}, \tag{44}$$

where

$$a_0 = \frac{h_0^3 - h_c^3}{h_0(h_1 - h_0)(h_0 - H)}, \quad a_1 = \frac{h_1^3 - h_c^3}{h_0(h_1 - h_0)(h_1 - H)}, \quad A = \frac{h_c^3 - H^3}{h_0(h_0 - H)(h_1 - H)}; \tag{45}$$

whence

$$\mathrm{e}^{S_0\sigma/h_0} = \frac{(h_1 - h)^{a_1}}{(h - h_0)^{a_0}(h - H)^A}\,\mathrm{e}^{h/h_0}. \tag{46}$$

There is, of course, no definite shock thickness, but it is assumed (as usual in such cases) that for practical purposes, only the variation of h from $h_0 + \epsilon(h_1 - h_0)$ to

$h_1 - \epsilon(h_1 - h_0)$ is significant (and can be measured), where ϵ is a suitably small positive number. The distance over which this increase in depth takes place is taken as the measure of shock thickness. This distance is easily calculated from (46); assuming that ϵ is very small, it may be approximated as

$$\frac{h_0}{S_0}(a_0 + a_1)\ln\frac{1}{\epsilon}. \tag{47}$$

Fortunately, this definition of the thickness is not sensitive to changes in ϵ, and $\epsilon = 0{\cdot}05$ is adequate. As pointed out earlier the thickness is of the order h_0/S_0. It may be noted that if the depth is well in excess of the critical value, h_c may be neglected in (45), and $a_0 + a_1$ becomes a function of the strength $(h_1 - h_0)/h_0$ alone; a graph of this function is given in figure 4.

The steady profile solution for the more general case in which C and R are functions of h may be obtained similarly, but numerical integration is required. Equation (37) expressing v in terms of h is unchanged but

$$v_1 = C(h_1)\{S_0 R(h_1)\}^{\frac{1}{2}} \quad \text{and} \quad v_0 = C(h_0)\{S_0 R(h_0)\}^{\frac{1}{2}}$$

lead to modifications in (39); the profile is given by

$$\frac{dh}{d\sigma} = -S_0\frac{(hU - Q)^2 h/(C^2 R S_0) - 1}{h^3 - Q^2/g}. \tag{48}$$

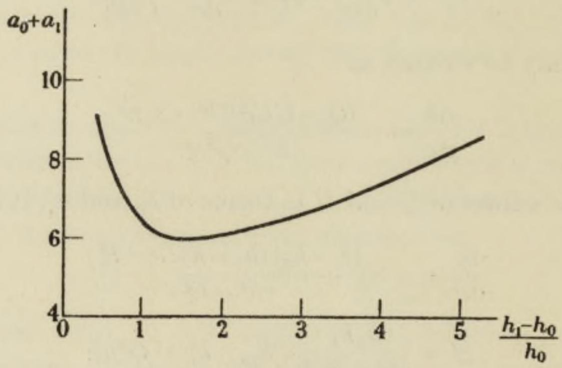

Figure 4

Keeping h_0 and h_1 fixed, we consider the form of (46) as F increases to and exceeds 2. Equally well, in view of the relation $F^2 = S_0 C^2/g$, we may interpret the variation in F as an increase in the value of the slope S_0, or a decrease in the friction coefficient f. In fact, it is convenient to study (42) and (46) through their dependence on h_c which is proportional to $F^{\frac{2}{3}}$. When $h_c < h_0$, the expected monoclinal wave is obtained. But when $h_c = h_0$, corresponding to $F = F_l$ say, the profile reaches the form shown in figure 5, with a finite slope at $h = h_0$. In a sense the wave is on the point of breaking; when $F = F_l$ is exceeded, (46) describes a curve of the form sketched in figure 6 and physical reality is lost. The critical value F_l is given from (39) and (43) by

$$h_c^{\frac{3}{2}} = F_l h_0 h_1 \frac{h_1^{\frac{1}{2}} - h_0^{\frac{1}{2}}}{h_1 - h_0} = h_0^{\frac{3}{2}},$$

that is,

$$F_l = \frac{h_0^{\frac{1}{2}}(h_1^{\frac{1}{2}} + h_0^{\frac{1}{2}})}{h_1}. \tag{49}$$

We observe that $F_l \leqslant 2$ (since $h_1 \geqslant h_0$), and $F_l = 2$ is reached only for $h_1 = h_0$. Two results have great significance in view of previous remarks. First, when $F = F_l$, the velocity of the wave is, from (38),

$$U_l = F_l g^{\frac{1}{2}} \frac{h_1^{\frac{3}{2}} - h_0^{\frac{3}{2}}}{h_1 - h_0}$$

$$= F_l \left\{ \frac{h_1}{(h_0^{\frac{1}{2}} + h_1^{\frac{1}{2}}) h_0^{\frac{1}{2}}} + 1 \right\} \sqrt{(gh_0)};$$

hence, substituting from (49), we have

$$U_l = F_l \left(\frac{1}{F_l} + 1 \right) \sqrt{(gh_0)} = v_0 + \sqrt{(gh_0)}. \tag{50}$$

Secondly, the maximum value of the slope, which is attained at $h = h_0$, is

$$\left(\frac{\mathrm{d}h}{\mathrm{d}\sigma} \right)_{h=h_0} = -\frac{S_0(h_1 - h_0)(h_0 - H)}{3h_0^2} = -\tfrac{1}{3} S_0 \left(\frac{h_1}{h_0} - 1 \right) \left(1 - \frac{h_1}{(h_0^{\frac{1}{2}} + h_1^{\frac{1}{2}})^2} \right).$$

After some manipulation this reduces, using (49), to

$$-\tfrac{1}{3} S_0 \left(\frac{2}{F_l} - 1 \right),$$

and the corresponding value of $\partial h / \partial t$ is

$$-U \frac{\mathrm{d}h}{\mathrm{d}\sigma} = \frac{1}{3} \frac{S_0}{F_l} (1 + F_l)(2 - F_l) \sqrt{(gh_0)}. \tag{51}$$

This is identical with the K of (24). Thus the steady profile wave (and this is the kinematic shock) has exactly the same speed as the dynamic wave-front (equation (50)), and its slope is exactly equal to the critical value which governs bore formation by dynamic waves. We may say then that at this value of F the two types of wave coalesce to move downstream together, and bore formation is imminent.

A remedy for the solution when F exceeds F_l is now clear. A bore must be fitted at the front of the wave increasing the depth discontinuously from h_0 to some value h^* on the profile (figure 6), to give a physically acceptable solution as shown in figure 7.

The equations expressing conservation of mass and momentum across the discontinuity are (Lamb 1932, p. 280)

$$Q^2 = \tfrac{1}{2} g (h_0 + h^*) h_0 h^*, \tag{52}$$

$$Q = (U - v_0) h_0 = (U - v^*) h^*, \tag{53}$$

where v^* is the particle velocity immediately behind the discontinuity. Equation (53) is already satisfied since $h = h^*$ is a point on the steady profile, and (52) remains to determine h^*. Since $Q^2/g = h_c^3$, the positive root of (53) is

$$\frac{h^*}{h_0} = \frac{\sqrt{\{1 + 8(h_c/h_0)^3\}} - 1}{2}. \tag{54}$$

Finally, it must be verified that this value for h^* exceeds the value $h = h_c$, the depth at which $\mathrm{d}h/\mathrm{d}\sigma$ becomes infinite in the solution represented by figure 6. But this is a fundamental property of bores; the flow ahead is subcritical and the flow behind is supercritical. Hence $h^* > h_c > h_0$. (The result may also be verified directly from (54).)

We see then that, as one expects, a bore will be formed whatever the Froude number of the initial flow, if the increase of discharge is great enough. As h_c increases further, the jump in height of the bore in this 'combined kinematic-dynamic shock' increases until at some value h^* becomes equal to $\bar{h}_1$ and the bore separates regimes of uniform flow. Before this value is reached, however, F exceeds 2, the flow becomes unstable and breaks down into a series of roll waves. Solutions representing roll waves have been obtained by Dressler (1949) as a series of bores separated by steady profile waves which satisfy (41); the solution is periodic and the whole configuration moves downstream at a steady speed. For the details, which do not directly concern us in the present work, the reader is referred to Dressler's paper.

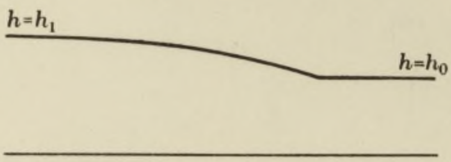

Figure 5

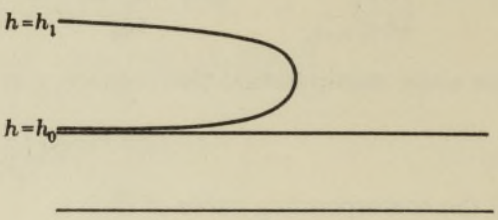

Figure 6

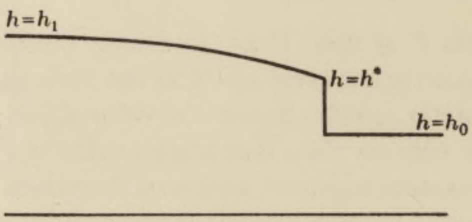

Figure 7

4. The kinematic wave theory of flood movement

In this section we set out the basic theory of floods treated as kinematic waves, with shock formation and prediction included as an essential part of the theory. Afterwards, improvements such as some allowance for diffusion effects in the kinematic waves may be added (see § 5). But the description of flood movement given in this section is the appropriate first approximation.

In general, the $q-k$ relation, from which kinematic waves are deduced, will depend upon the position x. The theory for the completely general relation will be described in this section, but first as an introductory example we consider a special form of the relation which has sound practical value and leads to mathematical simplification. This arises when the dependence of k on q and x is separable; that is,

$$k = a(x)f(q), \tag{55}$$

say. An example covered by this relation is the uniform river for which k is a function of q alone, and thus $a(x) = 1$; the inclusion of the additional factor $a(x)$ does not introduce any essential complication into the analysis, yet it greatly increases the practical value of the example. Thus, Seddon in his thorough investigation of flow conditions in the Missouri found that the variations of stage with discharge at different stations were related linearly. That is, if a station located at x_0 is taken as reference, the stage at position x is given by

$$h(x, q) = \alpha(x)\, h(x_0, q). \tag{56}$$

(It is unnecessary to add a further function of x to the right-hand side of (56), since by definition zero stage corresponds to the same value of q at x and x_0.) Hence the stage-discharge relation is separable. If, further, the breadth B is a function of x multiplied by a power of h (as, for example, in the cases of rectangular and triangular cross-sections), the concentration-discharge relation will be separable. Hence, observational results indicate some special interest in (55). From a theoretical point of view, it is a consequence of the simple Chézy or Manning formulae, when the slope S is a function of x alone and R is replaced by h. Then $v = C\{hS(x)\}^{\frac{1}{2}}$, so that with $q = vhB$, we have

$$C^2h^3B^2S(x) = q^2. \tag{57}$$

Thus, if C is a function of x alone (as in the Chézy formula) or the product of a function x with a power of h (as in the Manning formula) the relation between h and q is separable, provided B is again of the form $b(x)\, h^n$.

The equation of the characteristics, which represent in the (x, t) plane the paths of wavelets, assumes a simple form when the $q-k$ relation is (55). The velocity of propagation is $(\partial q/\partial k)_{x\,\mathrm{constant}} = \{a(x)\, f'(q)\}^{-1}$; hence the characteristics satisfy

$$\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)_{q\,\mathrm{constant}} = a(x)\, f'(q). \tag{58}$$

It is convenient to label a characteristic by the value of t when the wavelet passes a fixed observation point. If we choose this observation point as $x = 0$, and use T to denote the times at which the wavelets pass it, (58) integrates to

$$t = A(x)\, f'(q) + T, \tag{59}$$

where $A(x) = \int_0^x a(x)\,\mathrm{d}x$. Since q remains constant on a wavelet, q retains the value taken at $x = 0$ at the time T; therefore, we may set $q = q(T)$. If we suppose that this function $q(T)$ is known from observations made at $x = 0$, the flow is now determined—at least until shocks appear, with their attendant modifications of the flow. For, at any position x at the time t, the corresponding value of T with the appropriate value of q is determined so that (59) is satisfied. In the special case of the uniform river discussed in § 3, $a(x)$ may be taken as unity and $A(x) = x$; hence the characteristics are straight lines. More generally the characteristics are straight in the $(t, A(x))$ plane. The $(t, A(x))$ plane for a typical problem will be as in figures 2 and 3 with x replaced by $A(x)$.

If the disturbance observed at $x = 0$ were a decrease in discharge, then, since $f'(q)$ increases as q decreases,† successive wavelets leaving $x = 0$ would, according to (58), have smaller velocities and the flow would spread out smoothly with no

† Greater discharge always gives greater propagation speed (cf. § 2).

tendency to shock formation. In the (x, t) plane the characteristics diverge forming an 'expansion wave'. In this case, the above solution is adequate by itself to describe the flow. However, whenever $q(T)$ increases, the velocities of successive wavelets increase so that the earlier ones are ultimately overtaken by later ones, resulting in shock formation. This is represented by the convergence and eventual overlapping of characteristics in the (x, t) plane. When this occurs, the solution given by (59) does not give a unique value for q (since T is not unique when there is more than one characteristic through a given point (x, t)) and modifications must be introduced; a shock wave, changing the values of q and k discontinuously, must be fitted in. The shock is in fact a relatively narrow region in which, due to the relatively rapid change of q, the assumed $q-k$ relation becomes invalid. But, in the first instance, it may be treated as a discontinuous wave producing the appropriate abrupt changes in q and k; the more detailed behaviour of q and k in the shock region is represented by the steady profile solution of § 3 and can be included afterwards.

At each point of the shock, characteristics of the flow ahead and behind intersect as shown in figure 3. All the characteristics and the values of q on them are known; it only remains to determine where they are cut off and separated by the shock. It is obvious (graphically) that this will be achieved, if we can find how the pairs of characteristics which meet at the shock are related; in particular, a determination of the relation between the labels T_1 and T_2 of two such characteristics will suffice. This is now obtained.

With $T_2 > T_1$, the total flow across $x = 0$ between the times T_1 and T_2 is $\int_{T_1}^{T_2} q(T)\,\mathrm{d}T$. If t is the time at which the wavelets are at the same point x on the shock, then this quantity of fluid must flow out of the region between the wavelets by time t. Fluid passes a wavelet travelling with speed c, at a rate of $q-kc$; hence the *total* amount passing a wavelet is $\int_T^t (q-kc)\,\mathrm{d}t$. Since $c\,\mathrm{d}t = \mathrm{d}x$, we may write this in the alternative form $\int_0^x (q/c-k)\,\mathrm{d}x$; in view of (58) it is evaluated as $\{qf'(q)-f(q)\}\,A(x)$. Thus the required expression of continuity becomes

$$\int_{T_1}^{T_2} q(T)\,\mathrm{d}T = -[qf'(q)-f(q)]_{q_1}^{q_2}\,A(x), \tag{60}$$

where $q_1 = q(T_1)$ and $q_2 = q(T_2)$. The characteristic equations for the wavelets give

$$t = A(x)f'(q_1)+T_1, \tag{61}$$

$$t = A(x)f'(q_2)+T_2; \tag{62}$$

hence

$$A(x) = -\frac{T_2-T_1}{f'(q_2)-f'(q_1)}. \tag{63}$$

Equations (60) and (63) provide the required relation between T_1 and T_2. It is often convenient to introduce the excess of the discharge over its undisturbed value q_0, and write the left-hand side of (60) as $\int_{T_1}^{T_2}(q-q_0)\,\mathrm{d}T+q_0(T_2-T_1)$; from (63) this gives

$$\int_{T_1}^{T_2}(q-q_0)\,\mathrm{d}T = -[(q-q_0)\,f'(q)-f(q)]_{q_1}^{q_2}\,A(x), \tag{64}$$

in place of (60).

These shock equations are most easily dealt with when the flow on one side is the uniform flow $q = q_0$. This will ultimately be the case, for example, at the head of a 'hump' (as shown in figure 3), since even if the shock forms in the interior it eats its way through to head the flood. Then, q_1 takes the constant value q_0, and if T_f is the time of the first arrival of the disturbance at $x = 0$ (i.e. the label of the first characteristic of the disturbed flow), (64) becomes

$$\int_{T_f}^{T_2} (q-q_0)\,\mathrm{d}T = -\{(q_2-q_0)\,f'(q_2)-f(q_2)+f(q_0)\}\,A(x). \tag{65}$$

This relation between $q(T_2)$ and x, coupled with equation (62) for t, gives the equation of the shock path with T_2 as parameter. Equation (63) which would determine the corresponding value of T_1 is no longer required.

For the problem of a 'hump', wavelets are continually fed into the shock as time goes on; if T_l is the value of T for the last characteristic of the 'hump', $T_2 \to T_l$ and $q_2 \to q_0$ as $t \to \infty$. We may approximate (62) and (65) to give this asymptotic behaviour. For, when $q_2 - q_0$ is small.

$$\{(q_2-q_0)\,f'(q_2)-(f(q_2)-f(q_0))\} = \tfrac{1}{2}f''(q_0)\,(q_2-q_0)^2+O(q_2-q_0)^3.$$

Hence (65) is approximately

$$\int_{T_f}^{T_l} (q-q_0)\,\mathrm{d}T = -\tfrac{1}{2}f''(q_0)\,(q_2-q_0)^2\,A(x), \tag{66}$$

and we have the valuable result: *the increase in discharge at the shock is proportional to $A^{-\frac{1}{2}}(x)$*. Equation (62) becomes

$$t = A(x)\,f'(q_0)+(q_2-q_0)\,f''(q_0)\,A(x)+O(q_2-q_0)^2\,A(x)+T_l;$$

therefore the shock path is given by

$$t = A(x)\,f'(q_0)-A^{\frac{1}{2}}(x)\left\{-2f''(q_0)\int_{T_f}^{T_l} q(T)\,\mathrm{d}T\right\}^{\frac{1}{2}}+O(1).$$

In the special case of the uniform river for which $A(x) = x$, this is a parabola in the (x, t) plane; in general it is a parabola in the $(A(x), t)$ plane. We note that at the shock $A(x) \propto t$ to a first approximation, so that $q_2 - q_0 \propto t^{-\frac{1}{2}}$ and the width of the disturbed region increases proportional to $t^{\frac{1}{2}}$. As remarked in §3, the rate of decay like $t^{-\frac{1}{2}}$ is also obtained by linear theory.

When a disturbed flow on each side of the shock must be considered, the *implicit* relation between T_1 and T_2 cannot be avoided. Eliminating $A(x)$ from (63) and (64) it may be written

$$\frac{\int_{T_1}^{T_2} (q-q_0)\,\mathrm{d}T}{T_2-T_1} = \frac{[(q-q_0)\,f'(q)-f(q)]_{q_1}^{q_2}}{f'(q_2)-f'(q_1)}. \tag{67}$$

In general, determination of T_2 as a function of T_1 from this relation may be rather laborious. There is, however, an approximate form which offers great advantage in following the progress of the shock. The approximation applies rigorously when $(q_2 - q_1)/q_1$ is small, i.e. when the shock is weak. This will be true in the earlier stages after the shock is formed, and again for the ultimate decay of the shock; some correction may be needed between these two extremes (although the approximate results would still be of qualitative value). On the other hand, for such problems as the 'hump' (which is a typical one for floods), the shock soon moves to the head of the flood. Thus, for most of the motion, the (exact) description already given can be used; to supplement this, details of the shock near its formation may be sufficient.

In order to deduce the approximate form, we first remark that the right-hand side of (67) may be written

$$\frac{\int_{q_1}^{q_2} (q-q_0)\,\mathrm{d}f'(q)}{f'(q_2)-f'(q_1)}. \tag{68}$$

(The relation has a nice symmetry now, since it states that the mean values of $q-q_0$ with respect to T and with respect to $f'(q)$ are equal; this fact does not appear to help in the practical solution, however!) Then, if $(q_2-q_1)/q_1$ is small, (68) is $\frac{1}{2}\{(q_1-q_0)+(q_2-q_0)\}$ correct to the first order in $(q_2-q_1)/q_1$, and we have

$$\int_{T_1}^{T_2} (q-q_0)\,\mathrm{d}T = \tfrac{1}{2}(T_2-T_1)\{(q_1-q_0)+(q_2-q_0)\}. \tag{69}$$

On a graph of the function $q(T)-q_0$, the left-hand side of (69) is the area under the curve between the ordinates $T=T_1$ and $T=T_2$; the right-hand side is the area of the trapezium under the segment joining the points (T_1, q_1) and (T_1, q_2) of the curve. *Hence, the areas of the lobes* (*between the segment and the curve*) *on either side of the*

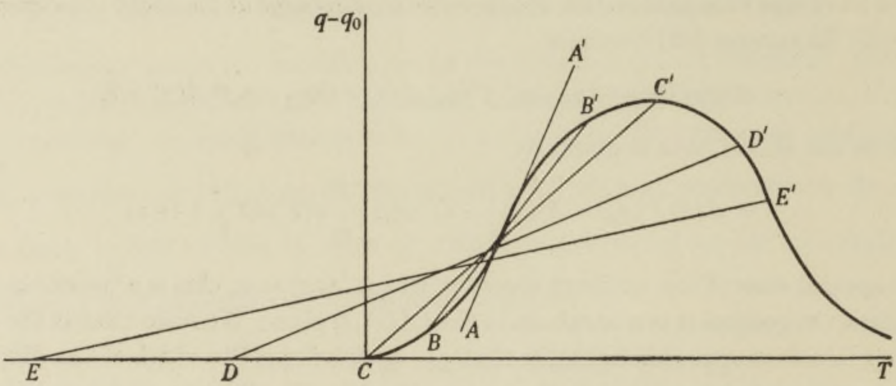

FIGURE 8

segment must be equal. Moreover, it should be noticed that in (63), $f'(q_2)-f'(q_1)$ will be approximately proportional to q_2-q_1 so that the slope of a segment is proportional to $1/A(x)$. Thus, *the slope of the segment approximately determines the corresponding position of the shock*. But it must be emphasized that this second approximation is only used as a rough guide in recognizing immediately the change in position of the shock from the change in slope of the segments; in any actual calculation, the value of x would be determined *accurately* from (63).

The progress of a shock after its formation can now be interpreted graphically by means of the segments which cut off lobes of equal area. We first describe the simple (yet most important) problem of a humped disturbance (figure 8). The position at which the shock first appears will correspond to the limiting case when the segment becomes the tangent, AA', at the point of inflexion of the curve. The values of q_1 and q_2 becomes equal, and since the slope of the segment is a maximum, the value of $A(x)$, and hence x, will be a minimum. As x increases, the corresponding segment decreases in slope. At first, the jump in q (the difference in the values at the end points of the segment) increases (BB', CC'), but it ultimately decreases (DD', EE') as $x \to \infty$, the segments tending to the axis. After CC' is reached, one end of the segment is on the axis, i.e. $q_1 = q_0$, and the flow on the upstream side of the shock is uniform. Then the exact determination of the shock (63) and (65) can be used instead of the present approximate method.

The point of shock formation is represented by the tangent AA'; hence, the shock forms on the characteristic specified by $T = T_i$, where T_i is the solution of $q''(T_i) = 0$ (with $q'(T_i) > 0$). The distance of the shock formation from the observation point $x = 0$ is found from (63) to be given by

$$A(x) = -\{f''(q(T_i))\, q'(T_i)\}^{-1}.$$

The subsequent motion and increase of strength of the shock very near its point of formation may be found *analytically* by approximating the relation between T_1 and T_2 for $T_i - T_1$ and $T_2 - T_i$ small. But eventually the full solution of (69) must be found. The above graphical interpretation is then valuable. After a little practice, the positions of the segments can be guessed fairly accurately and the details of the shock determined quite easily. In any case, such a guess provides a first approximation which can be checked by numerical evaluation of the expressions in (69). If tables of the functions $q(T) - q_0$ and $\int_0^T (q - q_0)\, \mathrm{d}T$ have been made, this initial estimate can be improved with little labour. When (69) is considered to be an insufficient approximation to (67), we suggest that it is simplest to make a first estimate by the 'segment method', and then adjust the values to satisfy (67) more accurately.

All graphs of the function $q(T) - q_0$ may be treated by the 'equal-area segments' in the way described above. There is one occurrence, however, which needs special mention, and which we illustrate by an example. Suppose there are two humps, corresponding to successive surges of the flood, as shown in figure 9. Shocks will be formed corresponding to each of the points of inflexion where the tangents have positive slopes; the tangents are shown as AA', $a'a$. The propagation of each shock will proceed, represented by segments such as BB' and $b'b$, respectively. But there may eventually be segments CC' and $c'c$ which have the same slope. If this occurs, the equality of slopes indicates that the corresponding x is the *same* for each shock, and since the characteristic corresponding to the common point c', C' meets both shocks, the time is also the same. *Thus the two shocks coalesce.* When this state of affairs is reached, all the points on the $q(T) - q_0$ curve between the two points of inflexion have been used for segments of one of the shocks. This means that all the wavelets represented by these points have been fed into one or other of the shocks; the remaining wavelets are represented by points on the curve either to the right of c or to the left of C. After the shocks combine a single shock continues, and it is represented by segments such as Dd; this part of the motion is described exactly as for the single hump.

Finally, in connexion with the segment method, it may be remarked that the approximate form (69) corresponds to the assumption that the shock velocity is the mean of the characteristic velocities of the flow on each side of it. (This result was proved in § 3 and is displayed as equation (40).) In the (x, t) plane this means that the shock line bisects the angle between characteristics which meet on it. This property is useful in picturing the (x, t) plane correctly and may even be used for a rapid (but rough) determination of the shock.

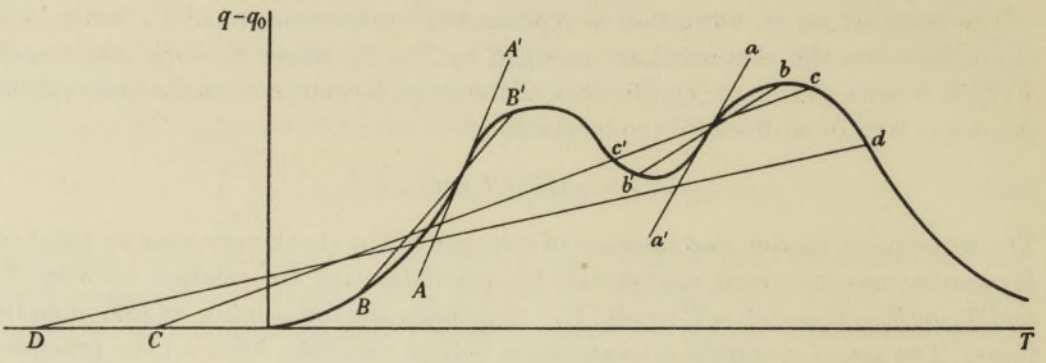

FIGURE 9

General $k-q$ relation

For the completely general relation $k = k(q, x)$, steps analogous to the above may be written down, but the practical solution of the various equations is more difficult. As before, q may be prescribed as a function of T alone, since it remains constant on a characteristic. The propagation speed is $c = (\partial q/\partial k)_x$ and the characteristics are

$$t = \int_0^x \left(\frac{\partial k}{\partial q}\right)_{x\ \text{constant}} \mathrm{d}x + T,$$

or

$$t = \frac{\partial V(q, x)}{\partial q} + T, \tag{70}$$

where $V(q, x) = \int_0^x k(q, x)\,\mathrm{d}x$ is the volume of water in the river between 0 and x. Evaluation of V, as a function of x for a range of values of q, may be lengthy, but at least it may be computed once and for all for any particular river; only the function $q(T)$ depends upon the particular flood. When $q(T)$ has been observed at $x = 0$, (70) provides the solution until shocks appear.

The determination of a shock again depends upon deducing the connexion between characteristics T_1 and T_2 which meet on it. Now, the total amount of fluid which passes a wavelet between 0 and x is

$$\int_0^x \left(\frac{q}{c} - k\right) \mathrm{d}x = q\frac{\partial V}{\partial q} - V.$$

Therefore, corresponding to (64) we have

$$\int_{T_1}^{T_2} q(T)\,\mathrm{d}T = -[qV_q - V]_{q_1}^{q_2}. \tag{71}$$

Further relations between t, x, T_1 and T_2 are provided by the characteristic equations,

$$t = V_q(q_1, x) + T_1, \tag{72}$$

$$t = V_q(q_2, x) + T_2; \tag{73}$$

in particular,

$$T_2 - T_1 = V_q(q_1, x) - V_q(q_2, x). \tag{74}$$

The latter equation may be used to write (71) in either of the alternative forms:

$$\int_{T_1}^{T_2} (q - q_0)\,\mathrm{d}T = -[(q - q_0)V_q - V]_{q_1}^{q_2}, \tag{75}$$

$$\frac{\int_{T_1}^{T_2} (q - q_0)\,\mathrm{d}T}{T_2 - T_1} = \frac{\int_{q_1}^{q_2} (q - q_0)\,\mathrm{d}V_q(q, x)}{V_q(q_2, x) - V_q(q_1, x)}. \tag{76}$$

In general, then, the shock determination depends upon the solution of two implicit equations (74) and (75) (say) for three quantities x, T_1, T_2. Without simplification their practical solution is not feasible. However, we may again consider the important case of uniform flow, $q = q_0$, ahead of the shock. Then T_f replaces T_1, and q_0 replaces q_1 in (75) to give the equation corresponding to (65). This provides an implicit relation between T_2 and x which may be solved numerically. Then the discharge behind the shock $q(T_2)$ can be found as a function of x, and the value t can be obtained from (73).

Apart from this case, we must again approximate (76) by (69), and describe the shock propagation by the segments cutting off lobes of equal area. This proceeds exactly as in the earlier discussion of the separable $k-q$ relation and further comment is unnecessary.

When the shock line has been obtained together with the values of q on each side of it, the final step is to replace the discontinuous shock by the appropriate monoclinal flood wave. Of course, since the latter has a *steady profile*, it applies strictly only to uniform flow conditions. Hence, mean values of the slope and the cross-section of the river must be taken in the neighbourhood of the shock, and these values used in applying the steady profile solution described in § 3. From the known values of q_1 and q_2, the corresponding values h_1 and h_2 of the depth may be found; then, for example, if the Chézy law is assumed, the required solution is given by (46). For other friction laws and general dependence of the hydraulic radius on depth, (48) must be integrated. This monoclinal flood wave is then centred on the shock line to provide details of the transition.

Tributaries and run-off

An important modification of the flood will occur when the waves pass a junction with a tributary. The discharge from the tributary will influence the flow in the main river upstream of the junction and produce modifications in the kinematic waves.

But we suggest that a good approximation to the effect on the flood will be obtained if the upstream influence is ignored, and as each kinematic wave passes the junction the discharge of the tributary is added to the value of q carried by the wave; the increase in q will produce a corresponding jump in the propagation speed. Although this method will give rise to a discontinuous increase in the stage, it is expected to be satisfactory provided that the tributary is appreciably smaller than the main stream. The discontinuity in stage would in fact be smoothed out by an adjustment in the flow upstream of the junction, and it is expected that the variation of stage would be rather like that given by the 'backwater curves' of steady flow. For a junction of streams of comparable size it may be necessary to take some explicit account of this.

The effect of run-off from the surrounding terrain will be similar except that the increase of the discharge will take place *continuously*. If the run-off has a volume flux $\mu(x,t)$ per unit length of the river, the equation of continuity must include μ as a source term, and we have

$$\frac{\partial k}{\partial t}+\frac{\partial q}{\partial x}=\mu(x,t).$$

Introducing $k=k(q,x)$, the kinematic waves will now be described by

$$\frac{1}{c}\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}=\mu(x,t).$$

The rate of change of q along a characteristic is then μ per unit distance, and in general the solution would be found by numerical integration along the characteristics. However, it is unlikely that the functional form of μ is known precisely, and the main interest is in deducing the effect of an estimated constant run-off. When μ is constant or a function of x alone, the equation for q may be solved explicitly (assuming that c is already known). For it may be written

$$\frac{1}{c}\frac{\partial}{\partial t}\left(q-\int\mu\,\mathrm{d}x\right)+\frac{\partial}{\partial x}\left(q-\int\mu\,\mathrm{d}x\right)=0;$$

hence, $q-\int\mu\,\mathrm{d}x$ remains constant on each characteristic. If $q=q(T)$ at $x=0$, then $q=q(T)+\int_0^x\mu\,\mathrm{d}x$ on the characteristic labelled by T. The characteristic itself is then given by

$$t=T+\int_0^x\frac{\mathrm{d}x}{c(q,x)}.$$

The case of a tributary is obtained when the run-off is concentrated at a point so that $\int\mu\,\mathrm{d}x$ is a step function, and the curvature of the characteristics (introduced by the run-off) is concentrated at a point.

5. Diffusion of kinematic waves

In the theory described in §4, diffusion is confined entirely to the interior of shock waves, where its effect is crucial in arresting the steepening of the kinematic wave profile. Outside these shock regions, diffusion is certainly small, but it may be valuable to include its effect as a second approximation. Mathematically, diffusion corresponds to the inclusion in (4) of an additional term proportional to a second derivative of q. This will arise if the flow-concentration relation involves, in addition to q, k and x, some dependence on a derivative of q or k. The existence of such a dependence is demonstrated by the well-known observational result that the stage-discharge rating curves for increasing and decreasing stage differ slightly. Thus, for example, we may assume that for each x, q is a function of k and $\partial k/\partial t$; alternatively, since $\partial k/\partial t = -\partial q/\partial x$, k would then be a function of q and $\partial q/\partial x$. If we take the latter form and substitute for k in the equation of continuity, we have

$$\frac{\partial k}{\partial q}\frac{\partial q}{\partial t}+\frac{\partial k}{\partial q_x}\frac{\partial^2 q}{\partial x\,\partial t}+\frac{\partial q}{\partial x}=0. \tag{77}$$

If, further, the coefficients of the derivatives of q in (77) are approximated as functions of q and x alone (by setting $q_x = 0$ therein, for instance), the equation may be written

$$\frac{\partial q}{\partial t}+c\frac{\partial q}{\partial x}+\nu\frac{\partial^2 q}{\partial x\,\partial t}=0, \tag{78}$$

where $c = \partial q/\partial k$ is the kinematic wave velocity as before, and $\nu = c\,\partial k/\partial q_x$. Equation (78) is typical of the equations representing the diffusion of kinematic waves. Other derivations may lead to one of the other second derivatives of q in place of $\partial^2 q/\partial x\,\partial t$ (in fact, the interchange can be carried out directly using the first approximation $\partial q/\partial t \simeq -c(\partial q/\partial x)$, but the consequences will be the same.

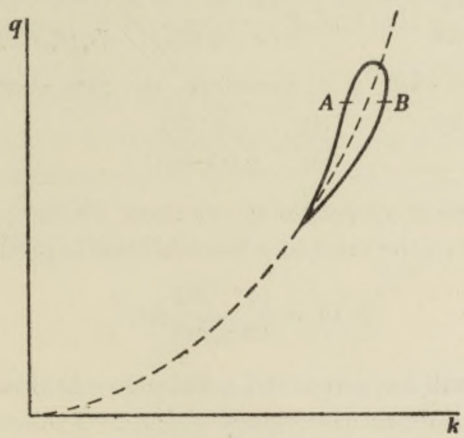

FIGURE 10

In practice the most satisfactory course may be to estimate a suitable form for the coefficient ν in (78) from the observational data for previous floods. For a given stage, the discharge is greater when the stage is rising than when it is falling, and the graph of q against k as a hump passes the observation point, is of the form shown in figure 10. To determine ν we must estimate $\partial k/\partial q_x$, which is the same as $-(\partial k/\partial k_t)_{q\,\text{constant}}$. If the two points on the curve corresponding to one value of q are denoted by A and B (see figure 10) then

$$\left(\frac{\partial k}{\partial k_t}\right)_{q\,\text{constant}} \simeq \frac{k_B-k_A}{(k_t)_B-(k_t)_A},$$

where subscripts A and B denote values at the points A and B. From the graph of k against time, the values of k_t can be found and therefore the value of ν deduced. The value of ν could be obtained for the different values of q to give ν as a function of q. The dependence of ν on q obtained in this way may not be too significant however, and it is probably sufficient to take a suitable average value for all q.

As in the case of the velocity c, values for ν may also be predicted on the basis of simple theories. Numerous approximate formulae have been suggested (see the article by Gilcrest in Rouse 1950) in attempts to described the deviations of the rating curves from the curve corresponding to steady flow conditions. The simplest of these is obtained by including the 'wedge storage term' in (11), i.e. S is taken to be $S_0 - h_x$ rather than S_0, where S_0 is the surface slope for uniform stage (the slope of the bottom for a uniform channel).

Then we may write
$$q = q^*(h,x)\sqrt{(1-h_x/S_0)}, \tag{79}$$
where $q^*(h,x)$ is the discharge for steady-flow conditions. This formula is of special importance since it is the starting point of the Forchheimer method for predicting the subsidence of a flood wave. Forchheimer considers the simplified problem of § 3, in which the variations of S_0, B and q^* with respect to x are neglected. Then, substituting (79) in the equation of continuity which may be written
$$B\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x} = 0,$$
we have
$$\frac{\partial h}{\partial t}+\frac{1}{B}\frac{\mathrm{d}q^*}{\mathrm{d}h}(1-h_x/S_0)^{\frac{1}{2}}\frac{\partial h}{\partial x}-\frac{q^*}{2BS_0(1-h_x/S_0)^{\frac{1}{2}}}\frac{\partial^2 h}{\partial x^2} = 0. \tag{80}$$
At the crest of the flood $\partial h/\partial x = 0$; therefore, the rate at which the height of the crest decreases is given by
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{q^*}{2BS_0}\frac{\partial^2 h}{\partial x^2}. \tag{81}$$
From observations of the flood profile at one time, $\partial^2 h/\partial x^2$ can be determined, and the change in the height of the crest at a time Δt later is predicted as
$$\Delta h = \frac{q^*}{2BS_0}\frac{\partial^2 h}{\partial x^2}\Delta t;$$
in this time, the crest will have reached a distance $c\Delta t$ downstream. In practice, it is more convenient to obtain the values of $\partial^2 q/\partial x^2$ from observational data and use $\partial^2 q/\partial x^2 \simeq (\partial q^*/\partial h)\,\partial^2 h/\partial x^2 = Bc\,\partial^2 h/\partial x^2$ to deduce $\partial^2 h/\partial x^2$.

If h_x/S_0 is neglected in the coefficients of equation (80), we have
$$\frac{\partial h}{\partial t}+c\frac{\partial h}{\partial x} = \frac{q^*}{2BS_0}\frac{\partial^2 h}{\partial x^2} \tag{82}$$
as an equation† representing the diffusion of kinematic waves, with $q^*/2BS_0$ as a coefficient of diffusivity. Alternatively, an equation of the form (78) may be deduced if (79) is modified by replacing h_x by $-h_t/c$. Then
$$q = q^*\sqrt{(1+h_t/cS_0)}; \tag{83}$$

† It may be noted that the linearized form of (82) is the same as (69) if $gh_0 - v^2$ is approximated by gh_0 in the latter.

this relation is known as the 'Jones formula'. Rouse (1950) states that (83) is a better approximation to reality than (79). Since $h_t = k_t/B$, (83) gives q as a function of x, k and k_t. We may then write it in the alternative form (using $q_x = -k_t$):

$$k = f\left(\frac{q}{\sqrt{(1-q_x/cBS_0)}}, x\right), \tag{84}$$

where $k = f(q, x)$ is the $k-q$ relation for steady flow conditions. Hence, finally, the coefficient ν in (78) is $q/2BS_0c$.

We now assume that (78) can be formulated either from observational data or from the above theoretical discussions, and turn to a consideration of its consequences in the theory of § 4. Previously, q was constant on each characteristic, but with the diffusion term we have

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} = -\nu\frac{\partial^2 q}{\partial x\,\partial t}, \tag{85}$$

so that q varies slightly, the rate of change depending on the values of q on neighbouring characteristics. For example (as in the Forchheimer method), where q is a local maximum so that $\partial^2 q/\partial x^2 < 0$, q will decrease along the characteristic, since $\partial^2 q/\partial x\,\partial t \simeq -\partial(c\,\partial q/\partial x)/\partial x > 0$. Conversely, when q is a local minimum it will have a tendency to increase. In this way diffusion smooths out the values of q, and this effect will be superimposed on the solution of § 4.

The introduction of a variation in q along the characteristics will also change the position of the characteristics in the (x, t) plane, since the propagation speed c depends on q. The changes in the values of c will be small; nevertheless, the total displacement of a characteristic may become large if the solution is continued far along it. However, in a first estimate for the correction due to diffusion, the characteristics could be left unchanged and determined as in § 4, but the variation of the values of q on them would be determined according to (85). For this purpose, it is convenient to take (85) in the form

$$\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{1}{c}\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} = \frac{\nu}{c^2}\frac{\partial^2 q}{\partial t^2}. \tag{86}$$

At $x = 0$, q is known as a function of t, and the integration of (86) along the characteristics can be carried out by the usual numerical methods. The simplest method is to evaluate the right-hand side of (86) from the data at $x = 0$, and deduce the increments in q at a distance Δx downstream along the characteristics from the formula

$$\Delta q = \left[\frac{\nu}{c^2}\frac{\partial^2 q}{\partial t^2}\right]_{x=0} \Delta x. \tag{87}$$

With the new values of q, the procedure can be repeated to furnish values of q at a further distance Δx downstream; this process is continued to extend the solution to points downstream. More refined schemes would include higher-order differences in the integration. Near a shock, the changes in q will become relatively large, and the integration is only continued until the values have been joined smoothly on to the steady profile solution which replaces the shock.

In some cases this adjustment of q on the existing characteristic network may be sufficient; it may even indicate that the unmodified solution of § 4 is adequate for the purpose in hand. But, in other cases, it may show the desirability of modifying the characteristics progressively with the values of q. The step-by-step procedure is easily adjusted to incorporate this, but the labour is increased. As before, $\partial^2 q/\partial t^2$ and c can be determined from the data at $x = 0$. Segments can be drawn, for the intervals Δx, in the characteristic directions (given by $\Delta t = \Delta x/c$); the increment of q along each of these segments is then given by (87). In this way q is obtained as a function of t at a distance Δx downstream. Repetition of this process continues the solution downstream. The shocks require special consideration, however. The above scheme would, in fact, include shocks automatically as regions of relatively large increases in q, since (78) is capable of providing steady profile shock solutions. But unless a special choice of ν is made, the shocks would not be described accurately; equation (78), particularly when obtained from theoretical considerations, has been introduced to estimate only *small* corrections to continuous kinematic waves. For example, q_x has been approximated by zero in the coefficients of (77) and this will not be valid inside the shocks. One method of describing the shocks accurately is to include them explicitly in the calculation. That is, when the presence of a shock is recognized (by the relatively large changes in q), a shock line should be drawn at each stage in the direction determined by the shock velocity $(q_2-q_1)/(k_2-k_1)$ appropriate to the flows which it separates. The final step would then be to centre on the shock line the steady profile solution which produces the required changes in q. In some respects, however, this method of determination is not as convenient as one in which the shocks are included automatically. We, therefore, investigate whether (78) may be artificially modified in order to give the shocks accurately. As mentioned above, (78) cannot really describe the flow inside a shock when c and ν are taken as functions of q and x alone. However, the dependence of ν on q may be so chosen that the shock will have the correct velocity; this is the main requirement. The shocks which will be described by (78) may be investigated by finding the steady profile solution. Taking q to be a function only of $\sigma = x - Ut$, we have

$$(c-U)\frac{\mathrm{d}q}{\mathrm{d}\sigma} = \nu U\frac{\mathrm{d}^2 q}{\mathrm{d}\sigma^2},$$

or on integration,

$$\int\left(\frac{c}{U\nu}-\frac{1}{\nu}\right)\mathrm{d}q = \frac{\mathrm{d}q}{\mathrm{d}\sigma}.$$

Now, if q_1 and q_2 are the values of q on the two sides of the shock, $\mathrm{d}q/\mathrm{d}\sigma$ must vanish for both $q = q_1$ and $q = q_2$; therefore

$$\int_{q_1}^{q_2}\left(\frac{c}{U\nu}-\frac{1}{\nu}\right)\mathrm{d}q = 0,$$

and thus U is given by

$$U = \int_{q_1}^{q_2}\frac{c}{\nu}\,\mathrm{d}q\bigg/\int_{q_1}^{q_2}\frac{1}{\nu}\,\mathrm{d}q.$$

We require that this be the accurate shock velocity $(q_2-q_1)/(k_2-k_1)$. Since $c = (\partial k/\partial q)^{-1}$, it will be so if ν is proportional to c. Of course, $\nu \propto c$ is not in agreement with the variation of ν outside the shock as given by (84), for example. But the variation of ν with q outside the shock would probably be ignored in any case and some constant value assigned to ν. Certainly, it is expected that observational data will only provide an approximate estimate for ν without showing its dependence on q. Thus, provided the constant of proportionality is adjusted to agree approximately with the value outside the shocks, we may take $\nu \propto c$. Then the shocks will be deduced by the step-by-step solution in their correct positions, without explicit consideration at each step.

Appendix

In this appendix, the details are given of the interpretation of (31) as a real integral involving the Bessel function $I_1(z)$. First, (31) is written as

$$Y = \mathrm{e}^{\xi(p+\frac{3}{2}\lambda)F}\{\mathrm{e}^{-\xi\sqrt{\{(p+\alpha)^2-\beta^2\}}} - \mathrm{e}^{-\xi(p+\alpha)}\}A(p) + \mathrm{e}^{-(1-F)\xi[p+\lambda(1-\frac{1}{2}F)]}A(p), \tag{88}$$

where $$\xi = \frac{x}{(1-F^2)\sqrt{(gh_0)}}, \quad \alpha = \lambda(1+\tfrac{1}{2}F^2), \quad \beta = \lambda(1-F^2)^{\frac{1}{2}}(1-\tfrac{1}{4}F^2)^{\frac{1}{2}}. \tag{89}$$

The second term in the expression for Y can be interpreted immediately as

$$\exp\left\{-\frac{\lambda x}{\sqrt{(gh_0)}}\left(\frac{1-\frac{1}{2}F}{1+F}\right)\right\}f\left(t-\frac{x}{v_0+\sqrt{(gh_0)}}\right), \tag{90}$$

since if† $f(t) = A(p)$, then

$$f(t-\mu) = \mathrm{e}^{-\mu p}A(p). \tag{91}$$

The interpretation of the first term can be deduced by the usual rules of Heaviside calculus from the known result (Doetsch 1947, p. 105) that

$$p(\mathrm{e}^{-\xi\sqrt{(p^2-\beta^2)}} - \mathrm{e}^{-\xi p}) = \frac{\xi\beta}{\sqrt{(t^2-\beta^2)}}I_1\{\beta\sqrt{(t^2-\xi^2)}\}H(t-\xi). \tag{92}$$

For, if $G(p) = g(t)$ then $pG(p+\alpha)/(p+\alpha) = g(t)\,\mathrm{e}^{-\alpha t}$; hence, the first term in (88) is

$$\mathrm{e}^{\xi(p+\frac{3}{2}\lambda)F}\frac{G_1(p)\,A(p)}{p},$$

where $G_1(p) = g_1(t) \equiv g(t)\,\mathrm{e}^{-\alpha t}$ and $g(t)$ is given by the right-hand side of (92). Now

$$p^{-1}G_1(p)\,A(p) = \int_0^t g_1(t-t')\,f(t')\,\mathrm{d}t';$$

hence, again using the 'shift rule' (91), the first term of (88) is

$$\mathrm{e}^{\frac{3}{2}\lambda F\xi}\int_0^{t+F\xi} g(t+F\xi-t')\,\mathrm{e}^{-\alpha(t+F\xi-t')}f(t')\,\mathrm{d}t'.$$

† As is customary, the equality sign is used rather loosely between functions and their operational representations.

On substituting for $g(t)$ from (95), and for ξ from (89), this becomes

$$\frac{\beta x\, e^{\frac{1}{2}\lambda F x/\sqrt{(gh_0)}}}{(1-F^2)\sqrt{(gh_0)}}\int_0^{t-\frac{x}{v_0+\sqrt{(gh_0)}}} \frac{I_1\left[\beta\sqrt{\left\{\left(t-t'-\dfrac{x}{v_0+\sqrt{(gh_0)}}\right)\left(t-t'+\dfrac{x}{\sqrt{(gh_0)}-v_0}\right)\right\}}\right]}{\sqrt{\left\{\left(t-t'-\dfrac{x}{v_0+\sqrt{(gh_0)}}\right)\left(t-t'+\dfrac{x}{\sqrt{(gh_0)}-v_0}\right)\right\}}}\, e^{-\alpha(t-t')} f(t')\,\mathrm{d}t'. \tag{93}$$

Finally, η is the sum of (90) and (93).

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